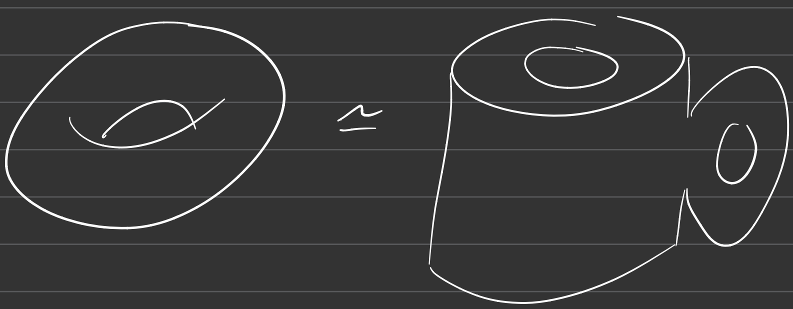
Topology

Continuous Maps

aka what we study



Topology

- Exam 75 %

- CA : 4 assignments 25 %

Advantage

Submit by Fridays +5 points

Submit by Monday -0 points

NO Extension

Topology

Geometry metric $\rightsquigarrow$ distances, angles ("smooth maps")

$\rightsquigarrow$ open sets ("continuous maps")

To define continuity of a map, the metric is not important. It can be defined in terms of open sets

Definition

A topology on a set X is a subset $\mathcal{T} \subset P(X)$ (power set)

(1) $\emptyset, X$ are in the topology

i.e. $\emptyset, X \in \mathcal{T}$

(2) Arbitrary unions of elements in $\mathcal{T}$ are in $\mathcal{T}$

(3) Finite intersections of elements in $\mathcal{T}$ are in $\mathcal{T}$

Let $S \subseteq \mathcal{T}$ Then $\bigcup_{s \in S} s \in \mathcal{T}$

Let $\mathbb{N} = X$

$\mathcal{T} \subseteq P(X)$. Let $\mathcal{T} = \{\{1\}, \{2\}, \ldots, \{1,2\}, \{2,3\} \ldots\}$

$=$ all finite subsets

$S = \{\{1\}, \{2\}, \{3\}, \{4\}, \ldots\} = \{\{i\} \mid i \in X\}$

$\bigcup_{s \in S} s = X$

Since $X \notin \mathcal{T}$, $\mathcal{T}$ is <u>not</u> a topology

Examples

(1) The trivial one

$\mathcal{T}_{trivial} = \{\emptyset, X\}$ for any set X

(2) The discrete Topology

$\mathcal{T}_{discrete} = P(X)$

(3) Let $X = \mathbb{R}$ and consider the collection of sets consisting of arbitrary unions of open intervals, ie

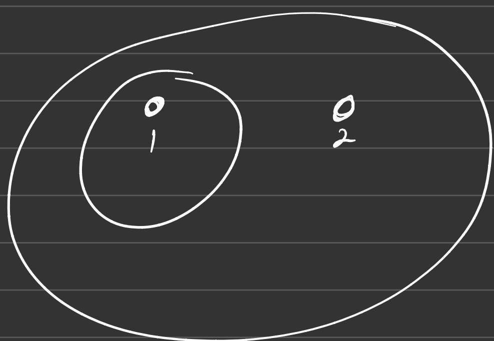
$$\mathcal{T} = \{ S \mid S = \bigcup_{\alpha \in I} (l_\alpha, r_\alpha) \}$$

is the standard topology on the real line

(4)

$X = \{0, 1\}$

$\mathcal{T} = \{\emptyset, X, \{1\}\}$



draw circles around the subsets which are in $\mathcal{T}$

(1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

(2) check by hand

(3) check by hand

This is the Serpinski Topology
(The "smallest" non-trivial / non-discrete one)

Definition

A topology $\mathcal{T} \subseteq \mathcal{P}(X)$, X some set st

(1) $\emptyset, X \in \mathcal{T}$

(2) $S \subseteq \mathcal{T}$, $\bigcup_{s \in S} s \in \mathcal{T}$

(3) $S \subset \mathcal{T}$, $|S| < \infty$, $\bigcap_{s \in S} s \in \mathcal{T}$

Why

Open set?

From metric spaces: A subset $S \subseteq \mathbb{R}^2$
st at every point $x \in S$, there
exists $B_d(x) \in S$ for some $d > 0$ is open

Claim

The open sets form a topology

(You likely saw f continuous

$\iff$ pre image of open set is open

"Proof"

Let $\mathcal{T}$ = open sets in $\mathbb{R}^2$

$\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$ (as open balls are subsets in $\mathbb{R}^2$)

- Unions?

Let $S \in \mathcal{T}$, so S = a collection of open sets

$x \in S \Rightarrow x \in S_i$ (one of the members of the collection)

$\Rightarrow \exists\, B_\delta(x) \subseteq S_i \subseteq \bigcup S_i = S$

- Intersections ✓

Open sets form a topology

So we may generalise the notion of continuity between functions of other spaces

Definition

Let $\mathcal{J}$ be a topology on X, $S \subseteq X$ is open if $S \in \mathcal{J}$

Examples (of topologies)

Particular point topology, Let $x \in X$

Define $\mathcal{J} = \{\emptyset\} \cup \{S \subseteq X \mid x \in S\}$

- $\emptyset \in \mathcal{J}$ ✓ $X \in \mathcal{J}$ ✓ $(x \in X)$
- union? Let $S \in \mathcal{J}$, say $S = \bigcup S_i \in \mathcal{J}$
Then either $x \in S$ or $S_i = \emptyset$

$\Rightarrow$ either $S_i = \emptyset \; \forall i$, or some $S_i \neq \emptyset$

$\Rightarrow \bigcup S_i = \emptyset$ $\qquad x \in S_i \subseteq \bigcup S_i$

$\Rightarrow \bigcup S_i \in J$

- intersection?

Let $S = \bigcap S_i$, $S_i \in J$

either $x \in S_i$ or $S_i = \emptyset$

if some $S_i = \emptyset \Rightarrow \bigcap S_i = \emptyset \in J$

if all $S_i \neq \emptyset \Rightarrow x \in S_i$ for all i

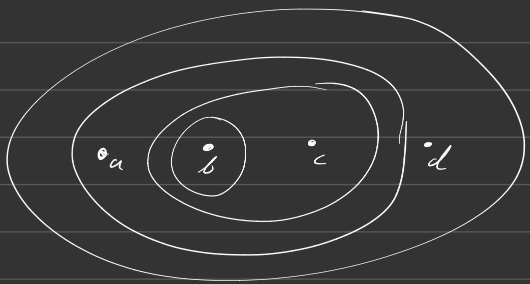
$\Rightarrow x \in \bigcap S_i \Rightarrow \bigcap S_i \in J$

(NB holds for arbitrary intersections)

Exercise B

Show J is closed under unions and intersections

$J = \{\emptyset, \{b\}, \{b,c\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\}$



Definition

A topology on X is metrizable if there exists a metric $d: X \times X \to \mathbb{R}_{\geq 0}$ st the collection of unions of open balls in the metric d form the given topology

Exercise

The discrete topology is metrisable

Proof

$d: X \times X \longrightarrow \mathbb{R}_{\geq 0}$ as

$$d(a,b) = \begin{cases} 1 & a \neq b \\ 0 & a = b \end{cases}$$

A ball $B_r(x) = \{ y \in X \mid d(x,y) < r \}$

$B_{\frac{1}{2}}(a) = \{a\} \Rightarrow$ ~~the~~ singletons are open balls

$B_2(a) = X \Rightarrow$ Given a set $S \subseteq X$,

$S = \bigcup_{s \in S} \{s\} \in$ in the topology (union of open balls)

What about the empty set?

$\Rightarrow$ discrete topology is metrizable

Is Ex B metrizable? NO!

Ex

co-cardinality topology (typically cofinite)

Let X be some set

$$\mathcal{T} = \{\emptyset\} \cup \{ S \subseteq X \mid X \setminus S \text{ is finite} \}$$

Check

$\emptyset \in \mathcal{T}$ ✓

$X \setminus X = \emptyset, \quad |\emptyset| < \infty$

- Unions

Let $S_i \in \mathcal{T}$ $i \in \mathcal{I}$

Then $X \setminus \bigcup S_i = \bigcap \underbrace{X \setminus S_i}_{\text{finite}} \subseteq X \setminus S_j$ for some j

$X \setminus \bigcap S_i = \bigcup X \setminus S$ (finite union of finite sets)

- Intersection

Bases of topology

Metric topology: A set in metric topology is open if it is an open union of open balls (intervals etc)

Idea of base of topology is to generalise this to all topologies

Definition

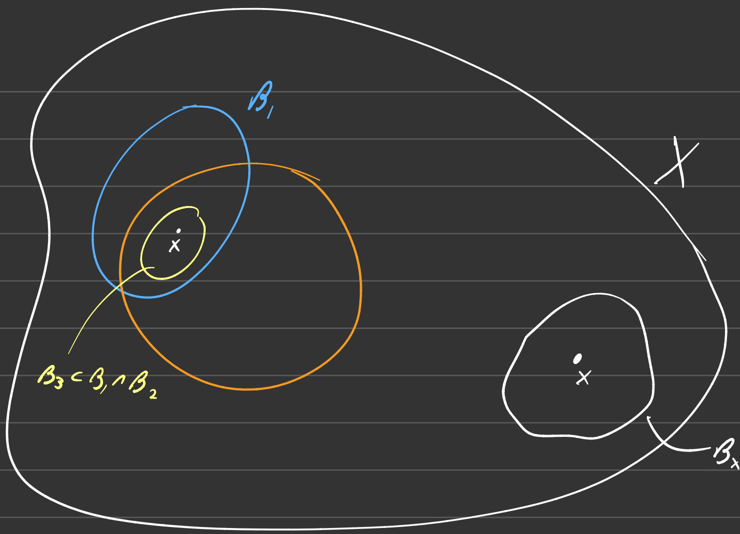
Base of a topology (does not need an actual topology), this is just a condition on sets.

Let X be a set and $\mathcal{B} \subseteq P(X)$. $\mathcal{B}$ is a base of a topology if

(1) $\forall x \in X$, $\exists \underbrace{B_x}_{\text{"basis element"}} \in \mathcal{B}$ st $x \in B_x$

(2) Let $B_1, B_2 \in \mathcal{B}$

$\forall x \in B_1 \cap B_2 \quad \exists B_3 \in \mathcal{B} \quad$ st $B_3 \subset B_1 \cap B_2$



Example

(1) $\mathcal{B} = \{X\}$ Boring and basic but Ian's favourite ex

Check

(1) Pick $x \in X$. Find B_x st $x \in B_x$, $B_x = X$

(2) Pick $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 = B_2 = X$

$\Rightarrow \forall x \in B_1 \cap B_2 = X, \ \exists B_x = X$

st $x \in B_x$

(2) $\mathcal{B} = \{ \underbrace{\{x\}}_{=B_x} \mid x \in X \}$

$$\underbrace{B_x \cap B_y = \{x\}}_{x=y} \text{ or } \underbrace{\emptyset}_{x \neq y}$$

(3) Arithmetic progression (base of a topology on $\mathbb{Z}$)

$S(a,b) = \{ an + b \mid n \in \mathbb{Z} \}$ $\quad (a \in \mathbb{N})$

(1) Pick $r \in \mathbb{Z}$

$r \in S(1,r) = \{ r + n \mid n \in \mathbb{Z} \} = \mathbb{Z}$

(2) Let $r \in \underbrace{S(a,b)}_{r = na+b} \cap \underbrace{S(a',b')}_{r = ma'+b'}$

$\Rightarrow r = na + b = ma' + b'$

Want $S(c,d)$ st $r \in S(c,d) \subset S(a,b) \cap S(a',b')$

Let $c = \mathrm{lcm}(a,a')$, $d = r$

$$\operatorname{lcm}(a,a')\gcd(a,a') = aa'$$

$$\operatorname{lcm}(a,a') = \frac{aa'}{\gcd(a,a')}$$

$$= a\underbrace{\left(\frac{a'}{\gcd(a,a')}\right)}_{\text{integer}}$$

$r \in S(\operatorname{lcm}(a,a'), r)$ ✓

$$\begin{gathered} S(\operatorname{lcm}(a,a'), r) \\ \shortparallel \\ \{r + n\operatorname{lcm}(a,a')\} \end{gathered}$$

$\boxed{(2)\ \exists\ B_3 \text{ st } x \in B_3 \subseteq B_1 \cap B_2}$

So we need to show that $B_3 \subseteq B_1 \cap B_2$

Sufficient to check $B_3 \subseteq B_1$

Let $t \in B_3 = S(c,d)$

so $t = r + \operatorname{lcm}(a,a')k$

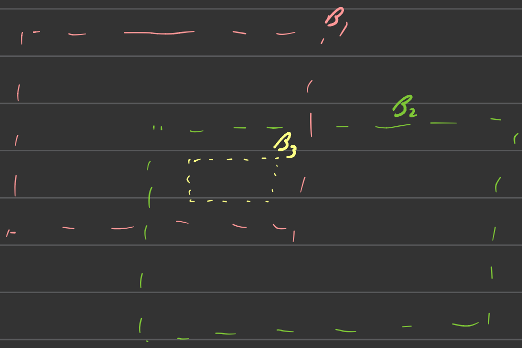
$$= na + b + a\left(\frac{a'}{\gcd(a,a')}\right)k$$

$$= a\left(n + k\frac{a'}{\gcd(a,a')}\right) + b \in S(a,b) = B_1$$

$\Rightarrow B_3 \subseteq B_1 \cap B_2$

(4) Open rectangles in dim 4

(1)  $\forall x \in \mathbb{R}^4$

(2) 

Definition and claim The topology generated by a base (of a topology)

Let $\mathcal{B}$ be a base on X. We define a topology, namely the topology generated by $\mathcal{B}$, as follows

$U \subseteq X$ open $\iff \forall\ x \in U$, there exists $B_x \in \mathcal{B}$ st $x \in B_x \subseteq U$ $(\star)$

Proof of claim

Lets call the subset of $P(X)$ of sets satisfying this condition $(\star)$, $\mathcal{J}$

(1) $\emptyset \in \mathcal{J}$ yes $(\forall\ x \in \emptyset, \dots)$

$X \in \mathcal{J}$ yes (condition (1) of base)

(2) Unions

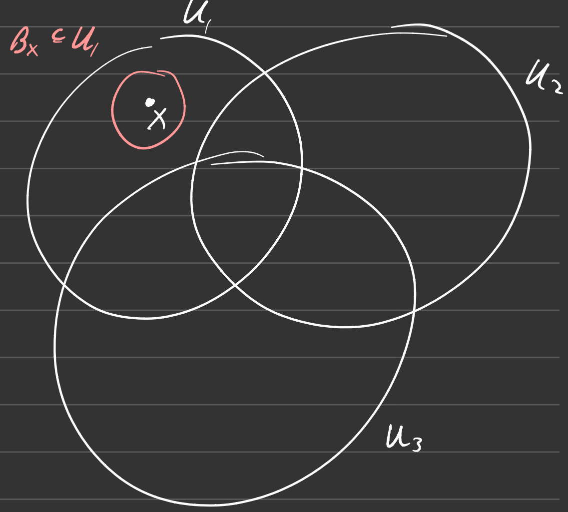
Let $U = \bigcup_{\alpha \in I} U_\alpha$, U_i satisfying $(\star)$

Let $x \in U$ WLOG $x \in U_1 \Rightarrow B_x$ st

$\Rightarrow \exists\ B_x$ st $x \in B_x \subseteq U_1$ by $(\star)$, so

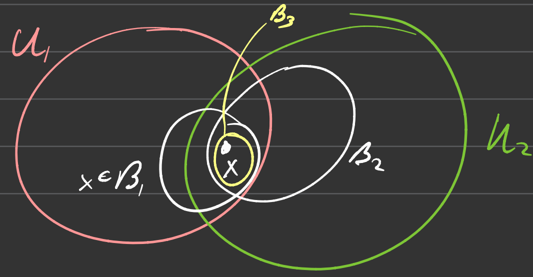
$x \in B_x \subseteq U_1 \subseteq \bigcup_{\alpha \in I} U_\alpha = U$

$\Rightarrow x \in B_x \subseteq U$



(3) Intersections

Let U_1, U_2 be satisfying $(\star)$



$\exists \ B_1$ st $x \in B_1 \subset U_1$

B_2 st $x \in B_2 \subset U_2$

$\Rightarrow B_3$ st $x \in B_3 \subset B_1 \cap B_2 \subseteq U_1 \cap U_2$

Theorem

The $\overbrace{\text{topology generated by a base}}^{\mathcal{T}}$ is a collection of all possible unions of basis elements Σ

"open sets are unions of open balls rectangles" (in $\mathbb{R}^2$)

Proof

$\mathcal{T} \subseteq \Sigma$, $\Sigma \subseteq \mathcal{T} \iff \mathcal{T} = \Sigma$

- ($\mathcal{T} \subseteq \Sigma$)

Let $U \in \mathcal{T}$. This means that for all $x \in U$, $\exists \ B_x$ st $B_x \in \mathcal{B}$ st $x \in B_x \subseteq U \Rightarrow \bigcup_{x \in U} B_x = U$

- ($\Sigma \subseteq \mathcal{T}$) Sufficient to show that $\forall \ B \in \mathcal{B}$, $B \in \mathcal{T}$

$\Leftrightarrow$ unions will also be in $\mathcal{T}$, by (2) of a topology

<u>Check</u>

$\forall\ x \in B$, does ~~there~~ exist

$B_x \in \mathcal{B}$ st $x \in B_x \subseteq B$?

$A: B_x = B$

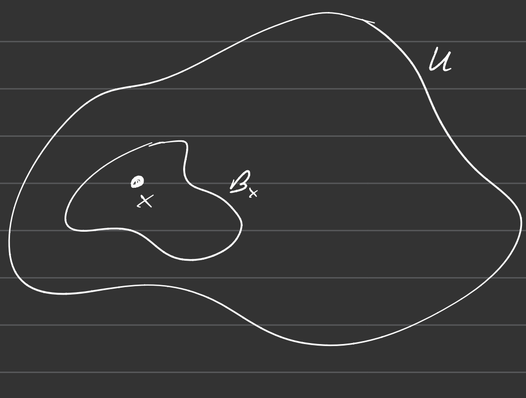
Base Identification Lemma

The topology generated by a base is the collection of unions of basis elements

Q Given a topology $\mathcal{T}$ on X, how to check that $\mathcal{B}$ generates the topology $\mathcal{T}$? (i.e. when is the topology generated by $\mathcal{B}$ equal to $\mathcal{T}$

A (basis identification lemma) Let $\mathcal{T}$ be a topology on X. If $\mathcal{B}$ is a collection of open subsets st for all U open in in $\mathcal{T}$, and all $x \in U$, there exists $B_x \in \mathcal{B}$, $x \in B_x \subseteq U$ $(\star)$,

(1) $\mathcal{B}$ is a basis



To check (a)

$\forall x \in X$

$\exists\ B_x \in \mathcal{B}$

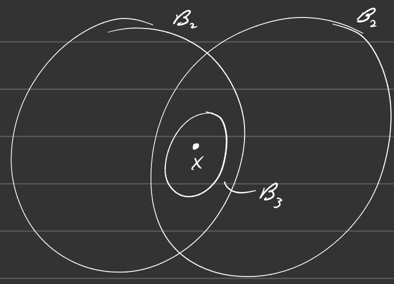
st $x \in B_x\ (\subseteq X)$

This is implied by $\star$ for $U = X$

(1) If $B_1, B_2 \in \mathcal{B}$ then

$\forall \ x \in B_1 \cap B_2$, there exists B_3 st

$x \in B_3 \subseteq B_1 \cap B_2$



As B_1, B_2 open

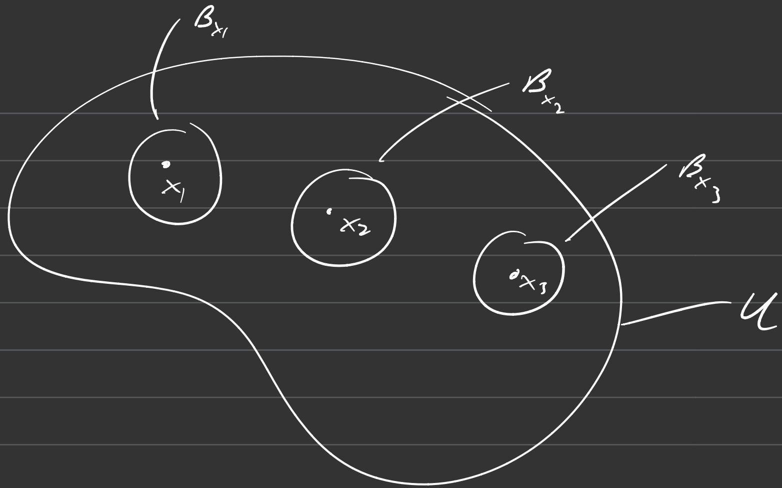
$\Rightarrow B_1 \cap B_2$ open

$\Rightarrow$ Apply (☆) to $B_1 \cap B_2$

(2) $\underbrace{\text{TOP gen by}}_{\substack{\text{collection of all unions} \\ \text{of basis elements}}} \mathcal{B} = \mathcal{T}$

As basis elements are open in $\mathcal{T}$ unions thereof are open in $\mathcal{T}$ so the topology generated by $\mathcal{B}$ is a subset of $\mathcal{T}$.

For the reverse inclusion let U be open in $\mathcal{T}$



$$Q: \bigcup_{x \in U} B_x = U \quad (\subseteq \text{ as } B_x \subseteq U)$$

$$\supseteq \ \forall x \in U, \ x \in B_x$$

$$\Rightarrow x \in \bigcup_{x \in U} B_x$$

Infinitely Many primes

$$S(a,b) = \{ a + nb \mid n \in \mathbb{Z} \}$$

$$a \in \mathbb{N}$$

<u>Last time</u>: The collection of such $S(a,b)$ forms a basis of a topology. Lets call this Fürstenberg topology

Assumption: We have finitely many primes

$S(a,b) = [b] \pmod a$ $\qquad [0]=[a]$ etc

We can choose $b \in \{0, \dots, a-1\}$

$$\Rightarrow \underbrace{S(a,0)}_{=[0]} \cup \underbrace{S(a,1)}_{=[1]} \cup \dots \cup \underbrace{S(a,a-1)}_{=[a-1]} = \mathbb{Z}$$

Fact

If $B \in \mathcal{B}$ then B is open in the topology generated by $\mathcal{B}$

$\Rightarrow S(a,b)$ are open in Fürstenberg topology

Definition

A set $A \subseteq X$ is closed if its complement is open.

$$S(a,b) = \mathbb{Z} \setminus S(a,0) \cup \dots \cup \widehat{S(a,b)} \cup \dots$$
$$\dots \cup S(a,a-1)$$

Assumption finitely many primes $p_1, \ldots, p_k$

Every integer other than $1, -1$ can be written as np_i for $n \in \mathbb{Z}$, p_i one of these primes

$\Rightarrow$ every integer is contained in $S(p_i, 0)$ for some p_i

$$\{1, -1\} = \mathbb{Z} \setminus \big(S(p_1, 0) \cup S(p_2, 0) \cup \ldots$$
$$\ldots \cup S(p_k, 0)\big)$$

<u>claim</u> $\{1, -1\}$ is open

$$|S(a, b)| = |\{na + b\}| = \infty$$

So if $\{-1, 1\}$ was open, then $S(a, b)$ generate the topology $\{-1, +1\}$ is a union of $S(a, b)$ which are infinite sets ↯ impossible

$\mathbb{Z} \setminus \bigcup S_i = \bigcap \mathbb{Z} \setminus S_i$

To show this:

$$\{-1, +1\} = \underbrace{(\mathbb{Z} \setminus S(p_1, 0))}_{\text{open}} \cap (\mathbb{Z} \setminus S(p_2, 0)) \cap \ldots \cap \underbrace{(\mathbb{Z} \setminus S(p_k, 0))}_{\text{open}}$$

$\boxed{\text{open}} \Leftarrow$ Finite intersection of open sets!

$S(p_1,0) \cup S(p_1,1) \cup \dots \cup S(p_1,p_1-1) = \mathbb{Z}$

$\Rightarrow \underbrace{\mathbb{Z} \setminus S(p,0)}_{\Rightarrow \text{ open!}} = \underbrace{S(p_1,1) \cup S(p_1,2) \cup \dots \cup S(p_1,p_1-1)}_{\text{open}}$

Order Topologies

Definition

A simple order on a set X is a relation $>$ satisfying

(1) either $x = y$ or $x > y$ or $y > x$

(2) $x > x$ is false

(3) $x > y$, $y > z \Rightarrow x > z$

Last time

Simple orders: • either $x < y$, $x = y$, $y < x$

• $x < x$ is false

• $x < y$, $y < z \Rightarrow x < z$

Definition

The order topology on X is the topology generated by the following basis elements

- (a,b) for $a<b$ $\qquad (a,b) = \{x \in X \mid a < x < b\}$
- $[a,b)$ if $a = \min(X,<)$
- $(b,c]$ if $c = \max(X,<)$

Example

Metric topology on $\mathbb{R}$ is generated by $\{(a,b) \mid a<b\}$, so the metric topology on $\mathbb{R}$ is the order topology wrt $<$ on $\mathbb{R}$

less then relation (a simple order)

Proof (of well definedness of definition)

We need to show that the collection of these sets forms a basis for a topology

(1) $\forall x \in X$, there exists a basis element containing x

3 cases

(a) $x = \min(X, <)$

$x \in [x, y)$ for $x < y$ or $x = y$

(b) $x = \max(X, <)$

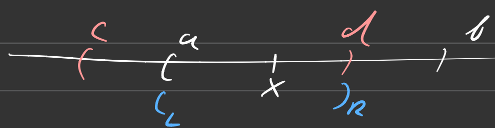
$x \in (y, x]$ for $y < x$

(c) neither (a) nor (b)

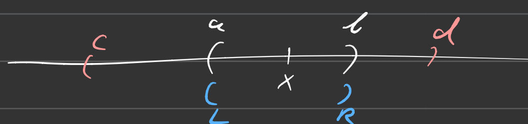
$\exists\ a, b$ st $a < x < b \Rightarrow x \in (a, b)$

(2) Suppose

$x \in (a, b)$



$x \in (c, d)$



Let $L = \max(a, c)$

$R = \min(b, d)$

$\Rightarrow x \in (L, R) \subseteq (a, b) \cap (c, d)$

$\Rightarrow (L, R)$ is the basis element for the 2nd condition to be a basis

$\square$

Example of order Topology

$X = \{1, 2\} \times \mathbb{N}$

elements : $(1,2), (2,1), (1,1), (1,3), \ldots$

$(a,b) < (c,d)$ if $a < c$ or $a = c$ and $b < d$

Is $(1,2) < (2,1)$ ✓

$$\begin{array}{ccccccc} (1,1) & < & (1,2) & < & (1,3) & < \\ \wedge & & \wedge & & \wedge & \\ (2,1) & < & (2,2) & < & (2,3) & < \end{array}$$

Q : What is the resulting topology
trivial

$\{(2,4)\} = \underbrace{(2,3), (2,5)}$

if $(2,3) < (x,y)$
$\Rightarrow \cancel{2<x}$ or $2 = x$

$(x,y) \in ((2,3), (2,5))$

$\Rightarrow x = 2$, and $3 < y < 5 \Rightarrow y = 4$

$\Rightarrow \{(2,4)\} = ((2,3),(2,5))$

$\Rightarrow$ The order topology cannot be $\{\emptyset, X\}$ as $\{(2,4)\}$ is open! $\Rightarrow$ NOT indiscrete

Q Discrete?

Are the singletons

Q: Is $\{(1,1)\}$ open?

$$\{(1,1)\} = [(1,1),(1,2)) \quad \checkmark$$

Q: Is $\{(2,1)\}$ open?

$$[(2,1),(2,2))$$

Not a basis element (as $(2,1) \neq \min(X,<)$!)

We need to show that $\{(2,1)\}$ is NOT a basis element $\Rightarrow$ it is not a union of basis elements as $|\{(2,1)\}| = 1$

$$\{(2,1)\} \neq [(1,1),(x,y))$$

If $(x,y) = (1,2) \Rightarrow$ not true

else $|[(1,1),(x,y))| > 1$

$\{(2,1)\} = ((a,b),(c,d))$? A: NO !

- If $a=2$ then $(2,x)<(2,1)$ is impossible
- If $a=1 \Rightarrow ((1,b),(2,c))$ contains more than 1 element !

So as $\{(2,1)\}$ is not open, the order topology is not discrete !

Product Topology

Let X, Y be topological spaces (ie a set X + topology on X)

Define a topology on $X \times Y$?

Definition

The product topology on $X \times Y$ is generated by

$\{U \times V \mid U$ open in X, U open in $Y\}$

Claim

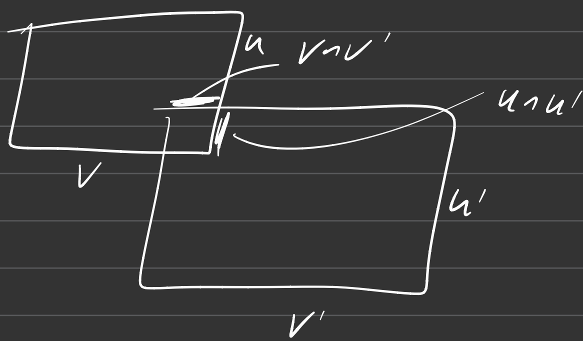
$\mathcal{B} = \{U \times V \mid U$ open in X, V open Y is a basis $\}$

(1) $(x,y) \in X \times Y \Rightarrow (x,y) \in X \times Y \in \mathcal{B}$

if $(x,y) \in U \times V \cap U' \times V'$

$$\Rightarrow (x,y) \in \underbrace{U \cap U' \times V \cap V'}_{\text{basis element}} \subseteq U \times V \cap U' \times V'$$

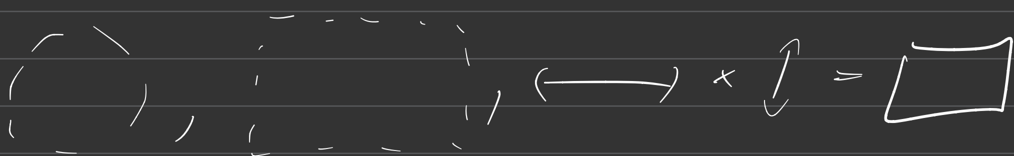
(2)



Claim

$$\{ B_X \times B_Y \mid B_X \in \mathcal{B}_X, \ B_Y \in \mathcal{B}_Y \}$$

generates the product topology on $X \times Y$ where $\mathcal{B}_X$, $\mathcal{B}_Y$ are basis for the topologies on X, Y



Product Topology

Given X, Y topological spaces,

$X \times Y$ = cartesian product w the topology generated by

$$\{ U \times V \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open} \}$$

Note that this is equivalent to the topology generated by

$$\{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$$

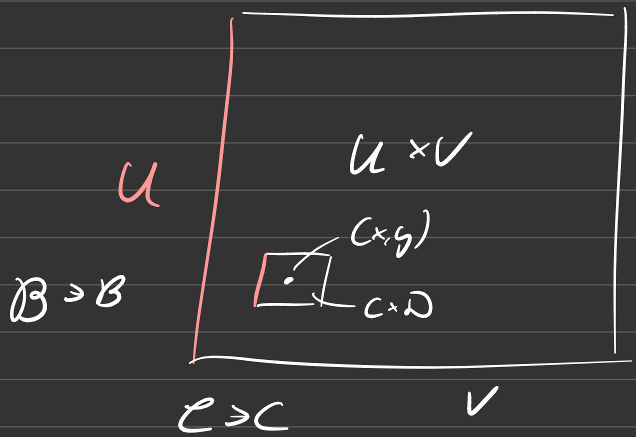
$\mathcal{B}$ = basis for the topology on X (ie given topology on X is generated by $\mathcal{B}$) and the the topology on Y is generated by $\mathcal{C}$

~~Proof~~ (Basis identification Lemma)

To check. For ~~all open sets~~ all basis elements $U \times V$, U open in X, V open in Y, O in $X \times Y$, and all $(x,y) \in \cancel{O}$ $U \times V$, does there exist a basis element D st $(x,y) \in D \subseteq O$

A: yes! because $\mathcal{B}$ is a basis for X, so ~~there~~ is some $B \in \mathcal{B}$ st $x \in B \subset U$. Similarly for y: Get $C \in \mathcal{C}$ $y \in C \subseteq V$

$\Rightarrow B \times C \subseteq U \times V$

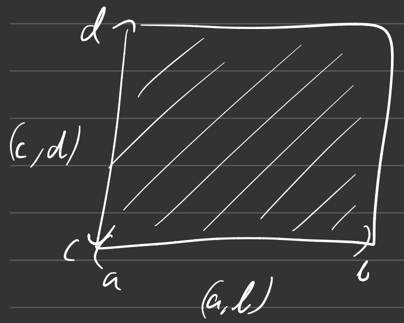


Example

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (standard topology on $\mathbb{R}$)

basis for $\mathbb{R}^2$: $\{B \times B' \mid B, B' \in$ basis elements for $\mathbb{R}\}$

$= \{(a,b) \times (c,d)\}$



$\Rightarrow$ using open intervals as a basis

$\Rightarrow$ basis elements are open rectangles

$\mathbb{R}^2$ as a metric space is generated by open balls, i.e.

$\{ B_r(x) \,/\, x \in \mathbb{R}^2, r > 0 \}$

is a basis

Is $\mathbb{R} \times \mathbb{R}$ as a space the same as $\mathbb{R}^2$ as a metric space? Yes!

Fact

If A, B are basis of topologies for X, and all basis elements of A are open in the topology generated by B, A

Then the topologies agree

To show that $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^2$ (metric) agree we use this fact

Proof

Lower limit Topology (Sorgenfrey line)

Definition

We topologize $\mathbb{R}$ as follows let

$$\mathcal{B} = \{ [a,b) \mid a < b \}$$

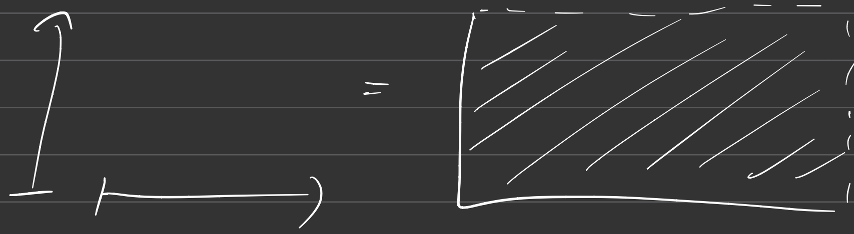
and let the Sorgenfrey line be generated by ~~the~~ basis

Note

$$(a,b) = \bigcup_{a<x<b} [x,b)$$

$\Rightarrow$ every set open in the standard topology on $\mathbb{R}$ is open in the Sorgenfrey line

$\mathbb{R}_\ell \times \mathbb{R}_\ell$ = topological space with base rectangle of the following form



(Bottom and left edges included)

This gives the <u>Sorgenfrey plane</u>

Subspaces

Let X be a topological space and $Y \subseteq X$ a subset

Definition

The subspace topology on Y is given by

$$\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T} \}$$

where $\mathcal{T}$ is the topology on X

<u>Exercise</u>: This is a topology

Example

$\mathbb{R} \cap [0,1]$ (order topology on $\mathbb{R}$)

claim

A basis for the subspace topology on $[0,1]$ is given by

$\{(a,b) \cap [0,1] \mid a < b\}$

$a < b < 0 \Rightarrow \emptyset$

$a > 1 \Rightarrow \emptyset$

If $a < 0$, $b \in (0,1)$

$\Rightarrow (a,b) \cap [0,1] = [0,b)$

If $0 < a < 1$, $b > 1 \Rightarrow (a,b) \cap [0,1] = (a,1]$

$0 < a < b < 1$

$\Rightarrow (a,b) \cap [0,1] = (a,b)$

$\Rightarrow 0 = \min([0,1], <)$

$1 = \max([0,1], <)$

Note that this collection ($[0,b), (a,1], (a,b)$) is up to including all of $[0,1]$ the basis for the order topology on $[0,1]$

so : Subspace of the order topology is the same as order topology of restricted order

Exercise : Find an example where this is finite

Fact

The product topology and subspace topology operations commute

so let $A \subseteq X$, $B \subseteq Y$ then the subspace $A \times B$ of $X \times Y$ is the same as the product of A as a subspace of X, B subspace of Y

Lemma

Let $\mathcal{B}$ be a basis for X, $A \subseteq X$. Then

$$\mathcal{B}_A = \{ B \cap A \mid B \in \mathcal{B} \}$$

is a basis for the subspace topology A

~~Proof of Fact~~

Check we can find one basis for both topologies

Subspaces

Let $Y \subseteq X$. The subspace Y is topologised as

$$\mathcal{T}_Y = \{ U \cap Y \mid U \text{ open in } X \}$$

ie U_Y open in $Y \iff U_Y = U \cap Y$
U open in X

- product topology operation "commutes" with subspace

ie $A \subseteq X$, $B \subseteq Y$ then

$A \times B$ top as ~~the~~ subspace of $X \times Y$

$= A \times B$ ——— product of subspaces

$A \subseteq X$, $B \subseteq Y$

~~Proof~~ Same basis for both

Closed Sets

Definition

Let $S \subseteq X$. We say that S is closed if $S = X \setminus U$ for U open (equivalently, $X \setminus S \in U$) is open)

Side Note

It is an easy exercise to define a topology in terms of closed sets

(1) $\emptyset, X$ closed

(2) arbitrary intersections of closed sets are closed sets are closed

(3) Finite unions of closed sets are closed

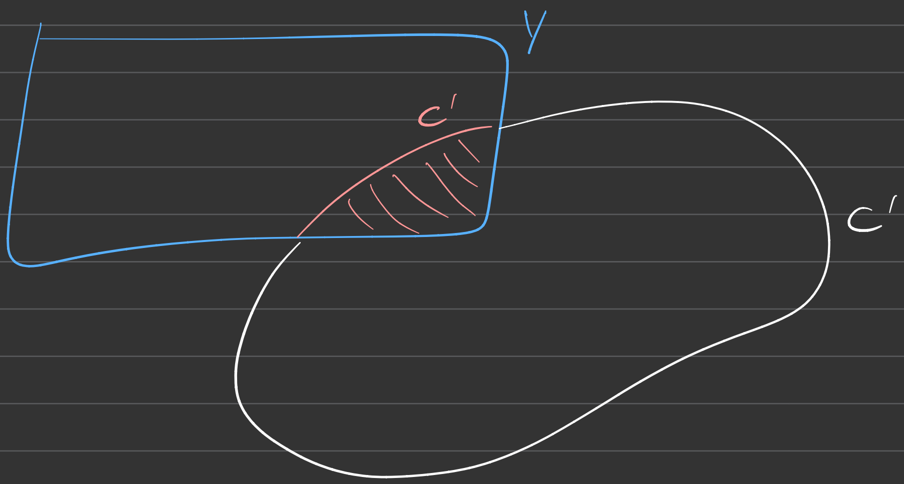
Notation

"$\leq$" subspace (ie topologised as subspace)

"$\subseteq$" subset (not considered as top space)

Lemma

Let $Y \subseteq X$. A set C is closed in Y if $C = C' \cap Y$ for C' closed in X.



Let C be closed in Y

$C = Y \setminus U$ ⊂ open in Y

$= Y \setminus (U' \cap Y)$ ← open in X

$\Leftrightarrow x \in Y,\ x \notin U' \cap Y$

$\Leftrightarrow x \in Y,\ x \notin U'$

$\iff x \in Y,\ x \in X \setminus U'$

$\iff x \in Y \cap (X \setminus U')$

$\Rightarrow Y \cap \underbrace{(X \setminus U')}_{\text{closed in } X}$

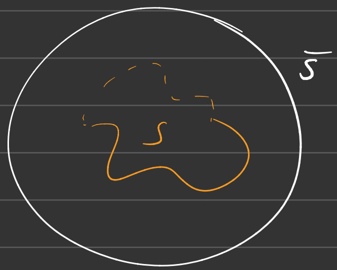
so let $C' = X \setminus U'$

Definition

Let $S \subseteq X$. We define

- closure of S,

$$cl_X(S) = \overline{S} = \bigcap_{\substack{C \text{ closed in } X \\ S \subseteq C}} C$$



"the smallest closed set containing S"

- interior of S,

$$int(S) = \mathring{S} = \bigcup_{\substack{U \text{ open} \\ U \subseteq S}} U$$

"largest open set contained in S"

Facts

$\bar{S}$ is closed, $\mathring{S}$ is open,

S is closed $\Rightarrow S = \bar{S}$

S is open $\Rightarrow S = \mathring{S}$

Proof Easy to prove

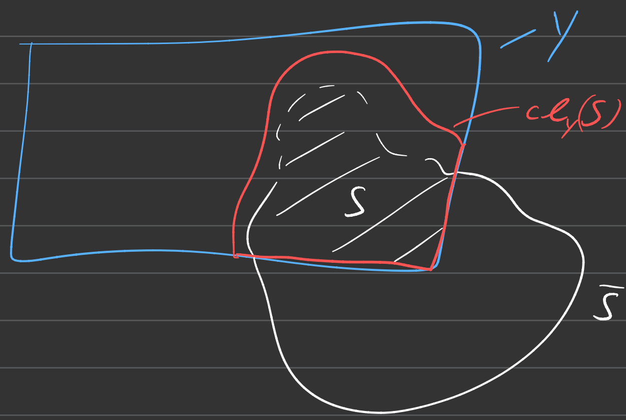
Lemma

Let $S \subset Y \subseteq X$. Then

$cl_Y(S) = cl_X(S) \cap Y = \bar{S} \cap Y$

Notation

$\bar{S}$ usually refers to $cl_X(S)$



Proof

$$cl_Y(S) = \bigcap_{\substack{C \text{ closed in } Y \\ S \subseteq C}} C$$

previous Lemma

$$= \bigcap_{\substack{C' \text{ closed in } X \\ S \subseteq C' \cap Y}} C' \cap Y$$

$$= \bigcap_{\substack{C' \text{ closed in } X \\ S \subseteq C' \text{ (as } S \subseteq Y)}} C' \cap Y$$

$$= \Big(\bigcap_{\substack{C' \text{ closed in } X \\ S \subseteq C'}} C'\Big) \cap Y$$

$$= \overline{S} \cap Y$$

Definition

A neighbourhood (nbhd) of $x \in X$ is an open U containing x

Lemma

The following are equivalent

(1) $x \in \overline{S}$

(2) every nbhd of x intersects S

(3) every closed set containing S also contains x

Proof

(1) $\Rightarrow$ (3)

$$x \in \overline{S} = \bigcap_{\substack{C \text{ closed} \\ S \subseteq C}} C$$

So if C contains S and is closed then $\overline{S} \subseteq C'$, so $x \in \overline{S} \Rightarrow x \in C'$

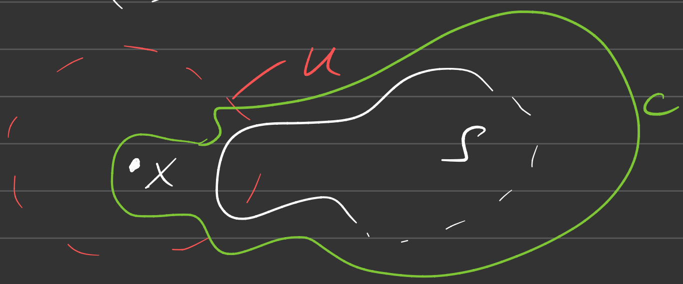
(3) $\Rightarrow$ (1)

Let $S \subseteq C$ and C closed implies $x \in C$

$$\Rightarrow \bigcap_{S \subseteq C,\, C \text{ closed}} C$$

contains x as all members of the intersection contain x

(2) $\Rightarrow$ (3)

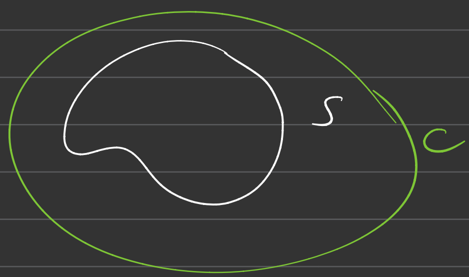


Suppose every neighborhood of x intersects S

Let C be a closed set containing S. We wish to show $x \in C$. If $x \notin C$, then $x \in X \setminus C$, an open set containing x is a nbhd of x

By assumption, $X \setminus C$ then intersects S. Since $S \subseteq C$, $S \cap (X \setminus C) = \emptyset$

So $X \setminus C$ cannot intersect S. So this is a contradiction



$\Rightarrow x \in C$

(3) => (2)

Let every closed set containing S also contain x. Let U be a nbhd of x. To show U intersect S. Let us assume this is not the case, i.e. $U \cap S = \emptyset$

Since $U \cap S = \emptyset$

$\Rightarrow S \subseteq X \setminus U$, so $X \setminus U$ is closed

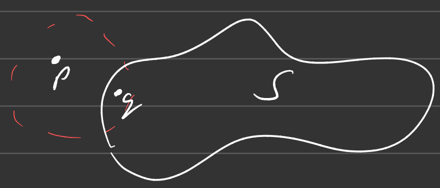
containing S, it must contain x.

But clearly $x \notin X \setminus U$ as $(x \in U)$

Definition

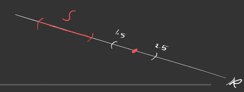
A limit point of a set $S \subseteq X$ in a top space X is $p \in X$ st every $\underbrace{\text{nbhd of } p}_{\text{open set containing } p}$ intersects S in a

point other than p. Write S' for ~~the~~ set of limit points of S



$q \neq p$

If true for all $U \Rightarrow p \in S'$



The following are equivalent

(1) $x \in \bar{S}$

(2) every basis element containing p (ie all nbhds of p which are basis elements) intersects S

<u>Proof</u> Exercise

The following are equivalent

(1) $x \in S'$

(2) All basis elements containing x intersect S in a point other than x

<u>Example</u>

Let $S = [0,1) \cup \{2\}$ and find S'

(1) Every point $[0,1]$ is a limit point

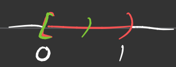
Recall Open balls centered at p form a basis of the metric order topology on $\mathbb{R}$, so we must check if all open balls centered at $p \in [0,1]$

intersects with S' contains a point other than p

$B_p(\varepsilon) = (p-\varepsilon, p+\varepsilon) \cap S = ($——•——$)$

<u>Case 1</u>

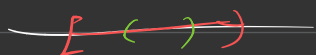
$B_0(\varepsilon) \cap S = [0, \varepsilon)$



<u>Case 2</u>

$B_1(\varepsilon) \cap S = (1-\varepsilon, 1)$

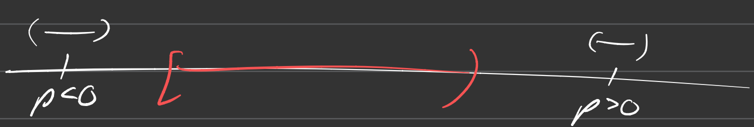
Case 3 $\cap$



(Have fun with subcases)

In all cases $(p-\varepsilon, p+\varepsilon) \cap S \setminus \{p\} \neq \emptyset$
so p is a limit point

$p = -2$ is not a limit point



$\Rightarrow$ no point in $\mathbb{R}$ other than $[0,1]$ is a limit point

($p = 2$ is not a limit point as
$(1.5, 2.5) \cap S \setminus \{2\} = \emptyset$

Example

$S = \{ \frac{1}{n} / n \in \mathbb{N} \}$ in $\mathbb{R}$

limit point 0

$(-\varepsilon, \varepsilon) \cap S = \{ \frac{1}{n} / \underbrace{\frac{1}{n} < \varepsilon}_{\infty \text{ many satisfies}} \}$

Archimedian prop of reals

$\Rightarrow |(-\varepsilon, \varepsilon) \cap S \setminus \{0\}| > 0$

$\Rightarrow 0$ is a limit point

No other point is a limit point

make radius small enough

Fact

$\bar{S} = S' \cup S$

Proof

$x \in \bar{S} \iff$ every nbhd of x intersects S

$\bar{S} \subseteq S' \cup S$:

every nbhd of a point $x \in \bar{S}$ intersects S either in a point other than x ($\Rightarrow x \in S'$)

or some nbhd, U, of x intersects S in x only

$\Rightarrow x \in U \cap S$

$\Rightarrow x \in S$

$\bar{S} \supseteq S' \cup S$

If $x \in S' \Rightarrow$ every nbhd of x intersects S

$x \in S \Rightarrow$ def^n

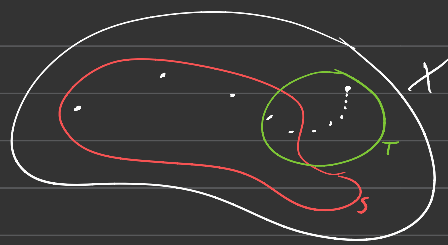
Sequences

Definition

A sequence in a top space X is $x : \mathbb{N} \longrightarrow X$, write $x_i = x(i)$

Definition

A set $S \subseteq X$ eventually absorbs a sequence x if S contains all but finitely many x_i

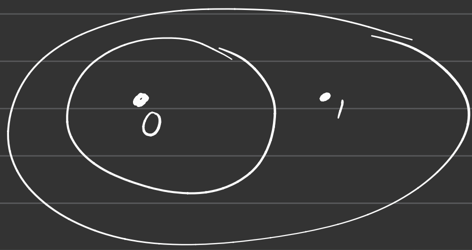


Definition

A sequence (x_i) converges to p if all nbhds of p eventually absorb (x_i)
(ie all but finitely many sequence elements lie in the nbhd)

Examples

Sierpinski topology



$x : \mathbb{N} \longrightarrow X$

$x_i = 0$

x_i converges to 1

Do all nbhds of 1 eventually absorb (x_i)? The only nbhd of 1 is X. Does X eventually absorb (x_i)? Yes! ∇

The sequence also converges to 0

check

Example

X with the indiscrete topology. Then every sequence converges to all point

$\mathcal{T} = \{\emptyset, X\}$

Sequences

$\mathbb{R}$ = metric top
$\mathbb{R}_\ell$ = lower limit top

Example

$x_n = (-1)^n \frac{1}{n} \, (\in \mathbb{R})$

in a metric ~~topology~~ on $\mathbb{R}$: $x_n \to 0$

- lower limit topology on $\mathbb{R}$
  (basis $[a, b)$ $a < b$ does not converge)

$x_n \to p \iff$ all basis elements containing p eventually absorb x_n i.e. all but finitely many terms lie in the basis element

$x_n \to 0$ in $\mathbb{R}_\ell$

Consider a basis element containing 0

$B_\varepsilon = [0, \varepsilon)$. claim B_ε does not absorb x_n

Elements in B_ε are non-negative, but $x_1, \ldots, x_3, x_5, x_{2i+1}$ are negative so $x_1, x_3, x_{2i+1} \notin B_\varepsilon \Rightarrow \infty$ many terms of (x_n) do not lie in B_ε so $x_n \not\to 0$

$$y_n = \frac{1}{n}$$

converges to 0 in both $\mathbb{R}$ and $\mathbb{R}_\ell$

Fact

If X is Hausdorff then sequences converge to at most to one point

Proof: Exercise

Category Theory

A Category consists of objects and morphisms between them

O	f
Objects	Morphisms
group	group hom
vec spaces	lin transform
rings	ring hom
algebra	algebra hom

$f : O \longrightarrow O_2$

$g : O_2 \longrightarrow O_3$

$g \circ f : O \longrightarrow O_3$ morphism

Continuous functions

Definition

A cts function $f : X \longrightarrow Y$ for top spaces X, Y is a map st for all open $U \subseteq Y$, $f^{-1}(U)$ is open in X

Fact

f is continuous if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$, a basis for ~~the topology~~ on Y

is a basis for the topology which generate the given ~~topology~~ on Y

Example

$f : \mathbb{R} \longrightarrow \mathbb{R}$ given $x \longmapsto x^2$

proof "soon"

Fact

Let (X,d) and (Y,e) be metric spaces $f:(X,d) \longrightarrow (Y,e)$ a map between metric spaces is continuous at a point $x \in X$ if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$

$$\boxed{d(x,x') < \delta \Rightarrow e(f(x), f(x')) < \varepsilon}$$

f is continuous if f is continuous at all points

$x' \in B_x(\delta) \Rightarrow f(x') \in B_{f(x)}(\varepsilon)$

$\Leftrightarrow f(B_x(\delta)) \subseteq B_{f(x)}(\varepsilon)$

Lemma

The following are equivalent

$\overline{A} = cl_X(A)$

(1) f is cts

(2) $\forall\ A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

(3) $\forall\ x \in X$ and all nbhds V of $f(x)$ there exists a nbhd U of x st $f(U) \subseteq V$

$B_{f(x)}(\varepsilon)$ $B_x(\delta)$

$f(B_x(\delta)) \subseteq B_{f(x)}(\varepsilon)$

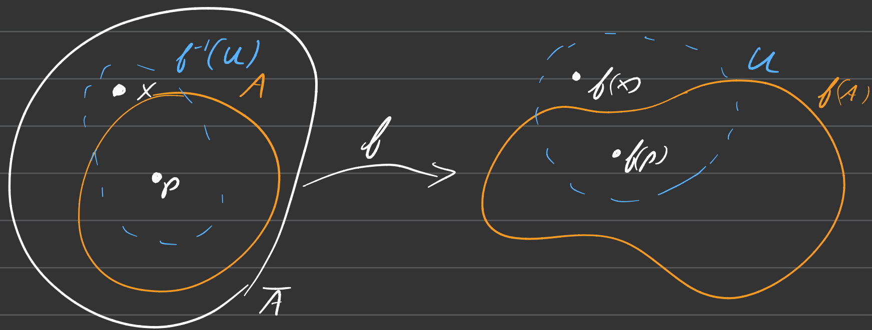
(4) for all closed $C \subseteq Y$ the pre-image $f^{-1}(C)$ is closed in X

For (3) I do not require that $f(U)$ is open

<u>Proof</u>

(1) $\Rightarrow$ (2)

Let $x \in \overline{A}$, To prove $f(x) \in \overline{f(A)}$



Recall $x \in \overline{S} \iff$ for all neighbourhoods U of x, $U \cap S \neq \emptyset$

$(p \in f^{-1}(U) \iff f(p) \in U)$

$f(x) \in \overline{f(A)} \iff \forall$ nbhds U of $f(x)$, $U \cap f(A) \neq \emptyset$ Consider $f^{-1}(U)$. (1) $x \in f^{-1}(W)$ (2) $f^{-1}(U)$ is open $\Rightarrow f^{-1}(U) \cap A \neq \emptyset$, if $= \emptyset$ then $f^{-1}(U)$ is a nbhd of x with trivial intersection with A, so $x \notin \overline{A}$

Say $p \in A \cap f^{-1}(U)$

$ff^{-1}(U) \subseteq U$

$\Rightarrow f(p) \in f(A) \cap U$

$\Rightarrow U \cap f(A) \neq \emptyset$

$\Rightarrow f(x) \in \overline{f(A)}$

Recall U was a nbhd of $f(x)$

$(2) \Rightarrow (4)$

Let $C \subseteq Y$ closed want to show $f^{-1}(C)$ is closed. Let $A = f^{-1}(C)$

To show $\overline{A} = A$ ($\Leftrightarrow$ A is closed)

is $\overline{A} \subseteq A$

$x \in \overline{A} \Rightarrow$ need to show $x \in A = f^{-1}(C)$

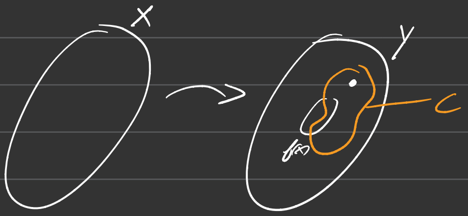
$\Leftrightarrow f(x) \in C$

(2) $x \in \overline{A} \Rightarrow f(x) \in f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(C))}$

$B \subseteq C$ $\subseteq \overline{C} = C$

$\Rightarrow \overline{B} \subseteq \overline{C}$

$f(f^{-1}(C)) \subseteq C$



(4) $\Rightarrow$ (1)

Let $U \subseteq Y$ be open

$\Rightarrow Y \setminus U$ closed

$f^{-1}(Y-U) = f^{-1}(Y) \setminus f^{-1}(U)$

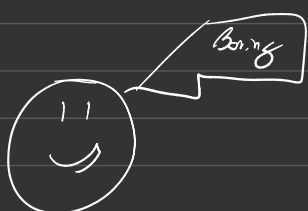
$= \underbrace{X \setminus f^{-1}(U)}_{\text{closed}}$

$\Rightarrow f^{-1}(U)$ open

(1) $\Rightarrow$ (3)

Pick $x \in X$. Pick nbhd V of $f(x)$ we need a nbhd U of x st $f(U) \subseteq V$

Let $U \subseteq f^{-1}(V)$

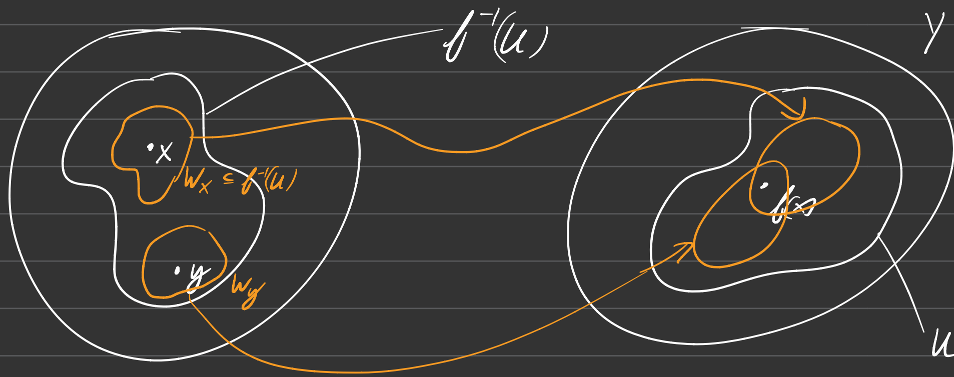


(3) $\Rightarrow$ (1)

Let $U \subseteq Y$ be open

<u>To show</u> $f^{-1}(U)$ is open

Let $x \in f^{-1}(U)$. Then $f(x) \in U$. U is a nbhd of $f(x)$ $\Rightarrow$ ~~there~~ exists a nbhd W of x st $f(W) \subseteq U$. $f(W) \subseteq U \Leftrightarrow W \subseteq f^{-1}(U)$



$$\Rightarrow \bigcup_{x \in f^{-1}(U)} W_x = f^{-1}(U) \quad (\text{as } x \in f^{-1}(U) \text{ and } x \in U_x)$$

$\Rightarrow$ Union of open sets

$\Rightarrow f^{-1}(U)$ is open

Facts

Let X, Y, Z be top spaces,

(1) $|f(X)| = 1 \Rightarrow f$ is cts

(2) $id : x \mapsto x$ is cts

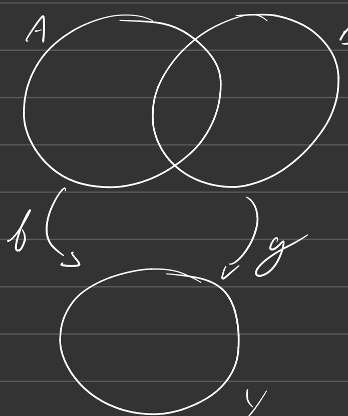
(3) $\underset{\text{cts}}{f} : X \to Y$, $\underset{\text{cts}}{g} : Y \to Z \Rightarrow g \circ f$ cts

(4) $A \subseteq X$ a subspace, $f : X \to Y$ cts

$\Rightarrow f|_A$ is cts

(5) $\pi_1 (X \times Y) \to X$, $\pi_2 (X \times Y) \to Y$ are cts

(6) $f : X \to Y \times Z$ is cts iff $\pi_1 \circ f$, $\pi_2 \circ f$ are cts



A, B closed

$f(x) = g(x)$

$\forall\ x \in A \cap B$

$\Rightarrow A \cup B \longrightarrow Y$ via $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

is cts if f, g are cts

f is a group hom is a group isomorphism

$\Leftrightarrow$ f bijective and f^{-1} is a group hom

Definition

f is a homeomorphism ("Isomorphism in Topology") if $f: X \longrightarrow Y$ is cts, f is a bijection, f^{-1} is cts

Definition

f is an embedding (if $f: X \to Y$ is cts, injective) and a homeomorphism onto its image

Definition

Let X be a top space and $A \subseteq X$ then A is dense if $\overline{A} = X$

Example

$\overline{\mathbb{Q}} = \mathbb{R}$ $\quad$ $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, algebraic reals

$\mathbb{Q} \setminus 0 = \mathbb{R}$

Definition

A is separable if it is dense and countable

X is separable if X admits a dense and countable subset

Definition

X is 2^{nd} countable if it admits a countable basis

Examples

$\mathbb{R}$ with basis $\{ B_q(\varepsilon) \mid q \in \mathbb{Q}, \varepsilon$ is of form $\frac{1}{n}$ for $n \in \mathbb{N} \}$

(Check)

$B_x(\varepsilon) = \bigcup$ balls in basis above

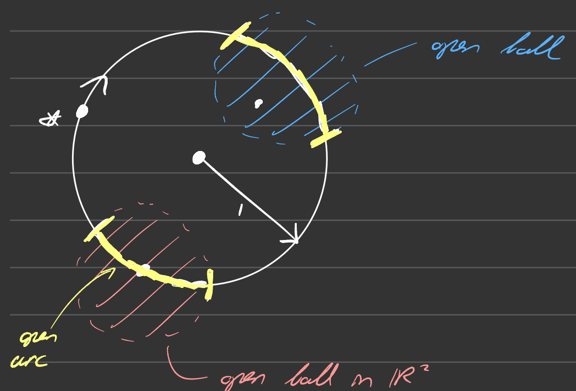
- $\mathbb{R}^n$
- Fact metric space 2^{nd} countable iff separable

Quotient Topology

$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$

topologized as a subspace of $\mathbb{R}^2$

Recall: A basis for S^1 is given by considering a basis of $\mathbb{R}^2$ and taking the intersection with S^1

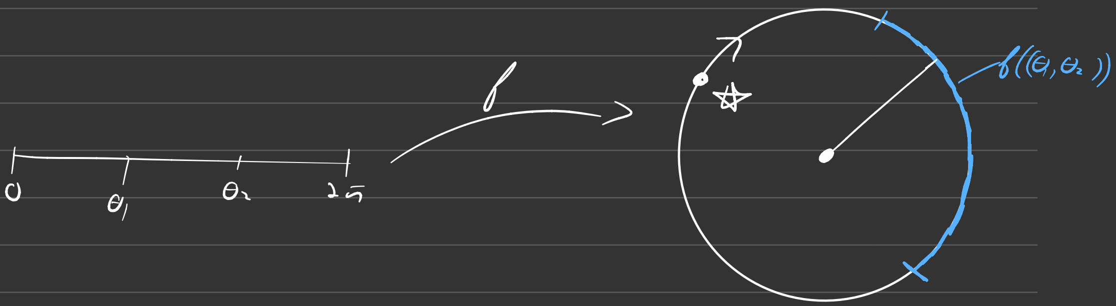


If we use open balls as our basis in $\mathbb{R}^2$ then the basis for S^1 are open arcs

Another option is to consider a map

$f : [0, 2\pi]$

measuring arc length along the circle from a basepoint ☆ in a specific direction the typical basis elements are of form $f((\theta_1, \theta_2))$



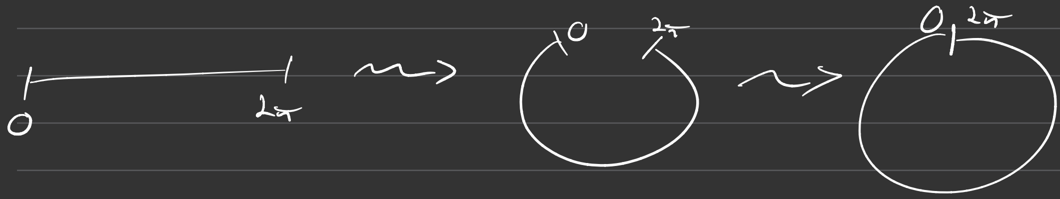
Goal: What conditions should a map f have so that we can reconstruct the topology on S^1 (without having to embed it in $\mathbb{R}^2$)

Note: f surjective

$$U \subseteq S^1 \text{ open} \iff f^{-1}(U) \subseteq [0, 2\pi] \text{ is open}$$

Note that f is almost a homeomorphism (injective except $f(0) = f(2\pi)$)

"Picture"



In this case we can check that giving

$$S^1 = \{(x,y) \mid x^2+y^2=1\} \quad \text{(as a set)}$$

the topology

$$U \subseteq S^1 \text{ open} \iff f^{-1}(U) \text{ open}$$

gives the topology on S^1 defined before

$[0, \theta_1)$ is open

$(\theta_2, 2\pi]$ is open as well

$\Rightarrow f^{-1}(U)$ open

Definition

A map $f: X \to Y$ is a quotient map if

(1) f is surjective (else $Y \setminus f(X)$ is discrete as $\emptyset$ is open)

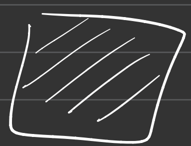
(2) $U \subseteq Y$ open $\iff$ $f^{-1}(U)$ is open

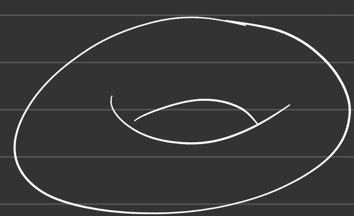
Example

$f: [0, 2\pi] \to S^1$

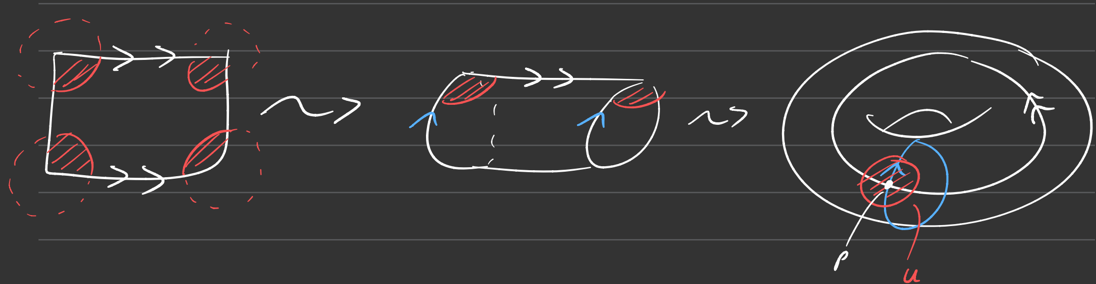
is a quotient map

Example





$[0,1] \times [0,1]$



Definition

Let X be a topological space and Y a set, p a surjective map $p: X \to Y$. The quotient topology on Y is the unique topology such that p is a quotient map. We say that Y is endowed with the quotient topology given by p.

(Specifically,

$U \subseteq Y$ is open $\iff$ $p^{-1}(U) \subseteq X$ is open)

Example

If we consider $f: [0, 2\pi] \to S^1$ as a set

then the quotient topology on S^1 agrees with the previously defined one.

Example

$X \to X/\sim$ (where $\sim$ is an equivalence relation on X)

We say that $X/\sim$ with quotient topology is an identification space

Example of Example

$X = [0, 2\pi]$

$0 \sim 2\pi$

$p \sim p$

$X/\sim \; = \{\{0, 2\pi\}, \{p\}\}$

$p \in (0, 2\pi)$

from before

Then f is really surj

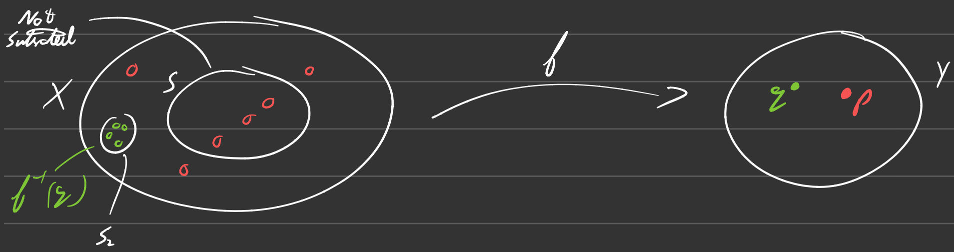
$f : X \longrightarrow X/\sim$

$y \longmapsto [y]$

Definition

A set S is saturated with respect to a map $f: X \to Y$ if

$$S \cap f^{-1}(p) \neq \emptyset \Rightarrow f^{-1}(p) \subset S$$



Fact

S is saturated $\iff$ S is a pre image of some set $U \subseteq Y$

Fact

A surjective continuous map is a quotient map iff maps open saturated sets to open sets

Fact

$p : X \to Y$, p cts, surjective is a $\underbrace{\text{quotient map}}_{\text{surjective} + U \subseteq Y \text{ open} \iff p^{-1}(U) \text{ open}}$

iff p maps $\underbrace{\text{saturated}}$ open set to open sets

A saturated

$\iff$

$p^{-1}(U) \cap A \neq \emptyset$

$\Rightarrow p^{-1}(Y) \subseteq A$

Definition

p is open (closed) iff p maps open (closed) sets to open (closed) sets

Fact

Open surjective cts maps are quotient maps

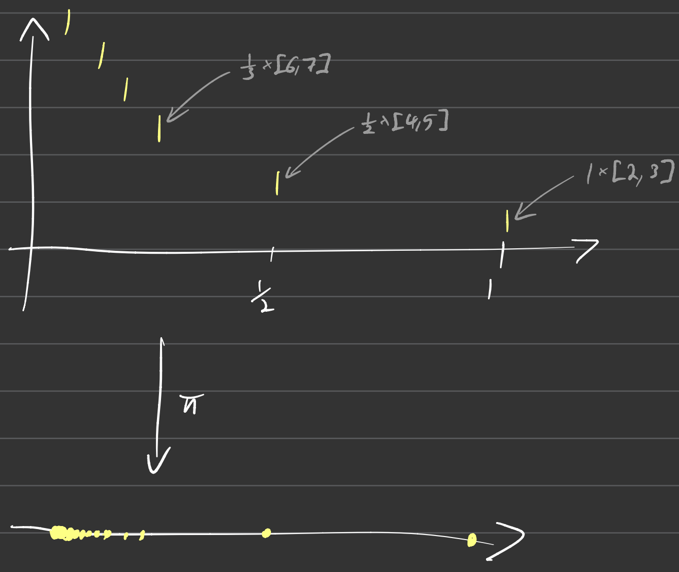
Closed ———

Projections

Claim

Projections ($\pi : X \times Y \longrightarrow X$) are open but not closed

$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is not closed. Consider
$S = \{ \frac{1}{n} \times [2n, 2n+1] \mid n \in \mathbb{N} \}$. this is not closed
but $\pi(S) = \{ \frac{1}{n} \} \subset \mathbb{R}$ is not closed



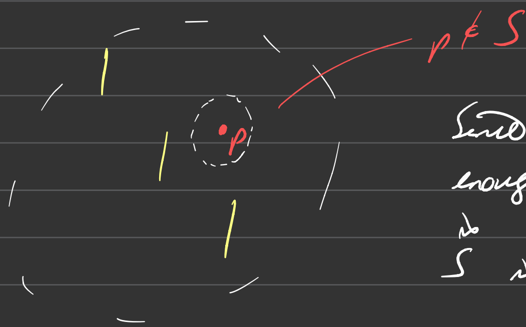
$\pi(S)$ is not closed

" $\{ \frac{1}{n} \mid n \in \mathbb{N} \}$ as 0 is a limit point but $0 \notin \pi(S)$

Any nbhd of 0 intersect $\pi(S)$

$\Rightarrow 0 \in \overline{\pi(S)}$ but $0 \notin \pi(S)$

S is closed



Since an open disc, small enough centered at p is disjoint from S, S is closed

$p : \mathbb{R} \longrightarrow \{1, -1\}$

$p(y) = +1$ if $y > 0$

$p(y) = -1$ if $y \le 0$

What is the quotient topology? i.e which topology makes p a quotient map

$\emptyset = p^{-1}(\emptyset) \iff \emptyset$ open

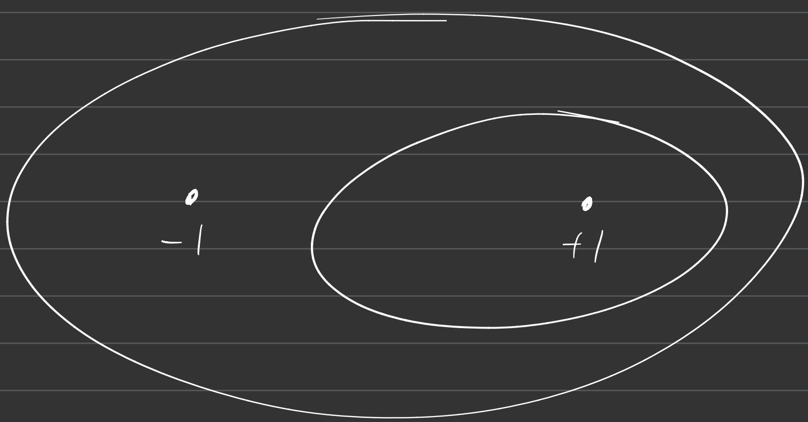
$p(\{-1\}) = (-\infty, 0] \iff \{-1\}$ not open

$\{+1\} = (0, \infty) \iff \{+1\}$ open

$p^{-1}(\{+1, -1\}) = \mathbb{R}$ open

$\iff$

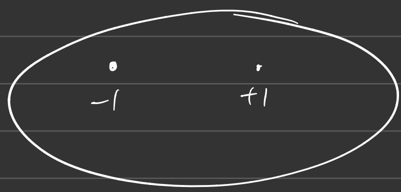
$\{+1, -1\}$ open



$$q(x) = \begin{cases} +1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \end{cases}$$

$$q^{-1}(\{+1\}) = \mathbb{Q}$$

$$q^{-1}(\{-1\}) = \mathbb{R} \setminus \mathbb{Q}$$

$\Rightarrow$ 

Fact

Let $p: X \to Y$, $A \subseteq X$. If p is a quotient map, A saturated with respect to p, then $p|_A : A \to p(A)$ is a quotient map if either of the following holds

- A open or closed
- p open or closed

Proof set theory (easy)

Maps out of quotient spaces

Fact

$p: X \to Y$ a quotient map $g: X \to Z$

a map constant on $\underbrace{\text{the fibres of } p}$

$x, y \in X$ st $p(x) = p(y)$

$\Rightarrow g(x) = g(y)$

$$\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "g"] & \\
y \in Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Then g induces a map f st $f \circ p = g$

(1) g quotient map iff f is a quotient map

(2) g cts iff f cts

$$\begin{tikzcd}
{[0, 2\pi]} \arrow[r, "p"] \arrow[r, bend right, "g"'] & S^1 \arrow[r, "f"] & Y
\end{tikzcd}$$

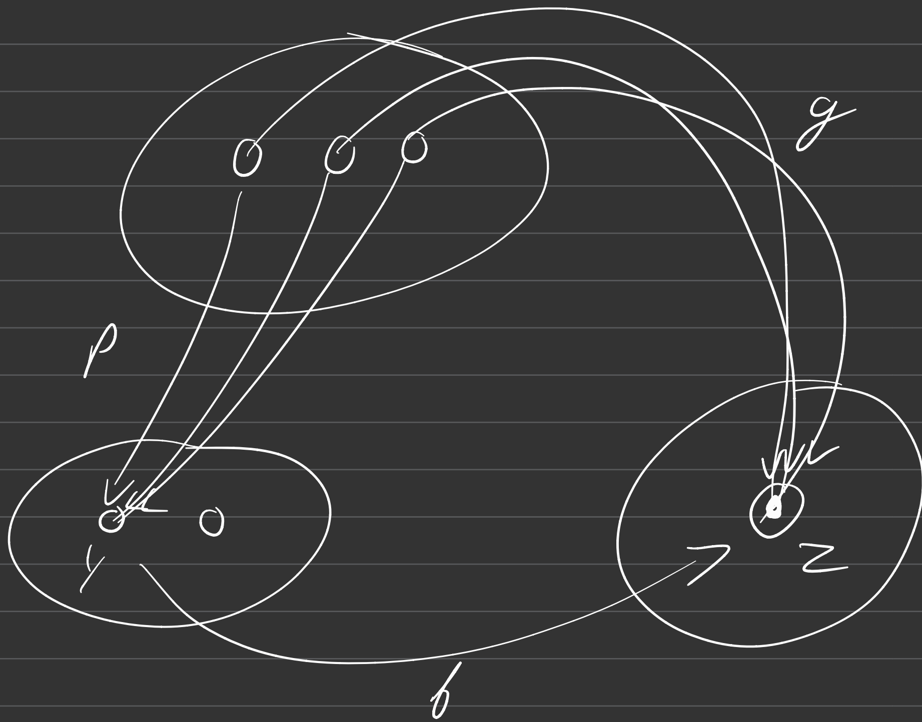
Proof

What is f?

define $f : Y \to Z$ st if $y \in Y$

then $g(p^{-1}(y)) = z$ (constant on fibers)

so define $f(y) = z$



claim g cts $\iff$ f cts

$g = f \circ p$

f cts $\Rightarrow$ g cts as composition of cts maps

g cts $\Rightarrow$ Pick $U \subseteq Z$, open. To show $f^{-1}(U)$ open

$g^{-1}(U) = p^{-1}(f^{-1}(U))$ $\quad (g = f \circ p)$

U open $\Rightarrow$ $g^{-1}(U)$ open

p quotient map

$\Rightarrow \underbrace{p^{-1}(f^{-1}(U))}_{= g^{-1}(U) \text{ open}}$ open iff $f^{-1}(U)$ open

$\Rightarrow f^{-1}(U)$ open

$\Rightarrow f$ cts

Lecture on

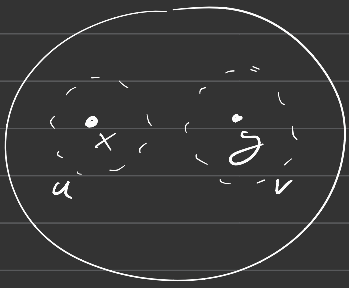
Wed 2-3pm

May 1

Separation Conditions (axioms)

Definition

X is Hausdorff if $\forall\, x \neq y$, there exists $U \ni x$, $V \ni y$ such that $U \cap V = \emptyset$



Implication

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y$$

where $x \in V$, $y \in U_y$

$$V \cap U_y \neq \emptyset$$

U_y, V are open

$\Rightarrow$ $X \setminus \{x\}$ is open

$\Leftrightarrow$ is closed

Definition

A space is T_1 if singletons are closed

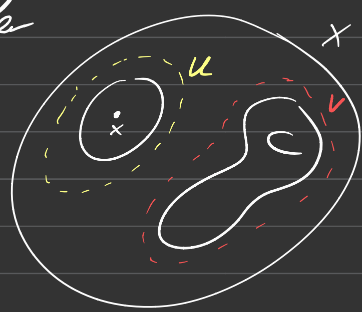
Fact

In a T_1 space (particularly HD), finite sets are closed. This implies that if $\mathcal{J}_c$ is the cofinite topology on X, and X is T_1 then $\mathcal{J}_c \subseteq \mathcal{J}_1$ the topology on X
"X refines the cofinite topology"

$\iff$ singletons are closed

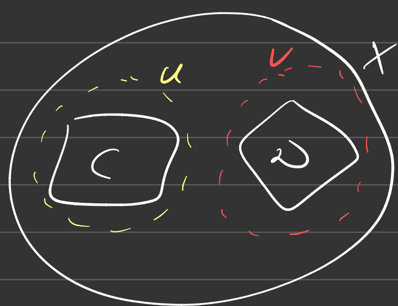
Definition

A T_1 space X is regular if $\forall x \in X$, all closed sets C not containing x there are open U, V
$U \cap V = \emptyset$, $x \in U$, $C \subseteq V$



Definition

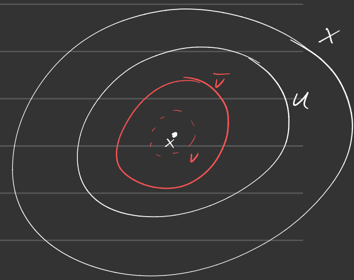
A T_1 space X is normal if $\forall$ closed disjoint sets $C, D \subseteq X$ there exists open sets $U, V \subseteq X$ s.t. $C \subseteq U$, $D \subseteq V$, $U \cap V = \emptyset$



Fact

Let X be a T_1 space

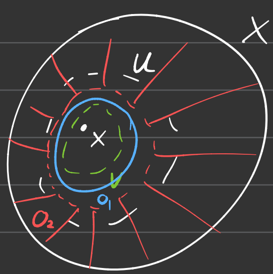
(1) X is regular iff for all x and for all nbhd U of x, there exists a nbhd V of x st $\overline{V} \subseteq U$



(2) X is normal iff for all closed sets C and all open U which contain C there exists an open set V st $C \subseteq V$ $\overline{V} \subseteq U$

Proof

X regular $\Rightarrow$ X satisfies the condition in (1)



Let $C = X \setminus U$ $(= U^c)$

X regular $\Rightarrow \exists\, O_1, O_2$ open such that $x \in O_1$, $C \subseteq O_2$, $O_1 \cap O_2 = \emptyset$

Let $V = O_1$ claim $\overline{V} \subseteq \overline{O_2} \subseteq U$

(Since O_2 is closed $\Rightarrow O_2^c \subseteq U$

$\Rightarrow \overline{O_1} \subseteq O_2^c \subseteq U$

Fact

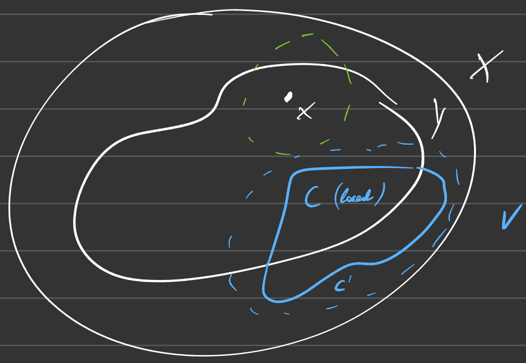
(1) A subspace of a Hausdorff space is Hausdorff

(2) A subspace of a regular space is regular

(3) Subspaces of normal spaces may not be normal

Proof

(2)



$C \subseteq Y$ closed

$\Leftrightarrow C = Y \cap C'$, C' closed in X

$\Rightarrow V \cap Y$, $U \cap Y$ open

in Y, disjoint $C \subseteq V \cap Y$, $x \in U \cap Y'$

(3)

Cannot guarantee existence C', D' disjoint such that $C \subseteq C'$ $D \subseteq D'$

Example

$\mathbb{R}_\ell$ is normal

~~Exercise~~

Fact

$\mathbb{R}_\ell \times \mathbb{R}_\ell$ Sorgenfrey plane) is not normal

Compactness

Definition

A cover for a topological space is a collection $\{U_\alpha\}_{\alpha \in I}$ of open sets such that $X \subseteq \bigcup U_\alpha$.

Definition

X is compact if every cover $\{U_\alpha\}_{\alpha \in I}$ admits a finite subcover ie

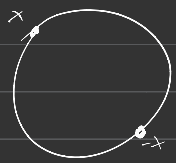
$$X = \bigcup U_\alpha \quad \alpha \in J \subseteq I \quad \text{st} \quad |J| < \infty$$

Example

is compact

Example

$$\mathbb{R}P^2 = D^2 \big/ x \sim (-x)$$



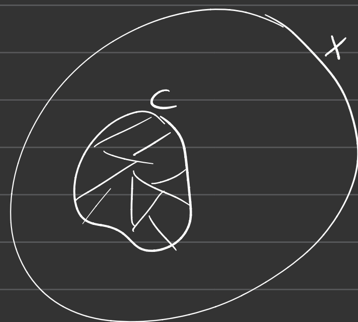
Fact

A closed subset of a compact space is compact

Claim

X compact, $C \subseteq X$ closed $\Rightarrow C$ compact

$C \subseteq \bigcup_{\alpha \in I} U_\alpha$, U_α open



Consider

$$X = \underbrace{X \setminus C}_{\text{open}} \cup \bigcup_{\alpha \in I} U_\alpha$$

$\Rightarrow X$ compact, finite subcover

$$\Rightarrow X = X \setminus C \cup \underbrace{U_1 \cup U_2 \cup \dots \cup U_n}_{\supseteq C}$$

$$\Rightarrow C \subseteq U_1 \cup \dots \cup U_n$$

$\Rightarrow \{U_\alpha\}_{\alpha \in I}$ arbitrary Compact

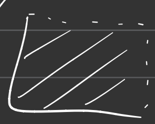
Definition

A space is Lindelöf if every cover

$$\bigcup_{\alpha\in I} U_\alpha = X$$

admits a countable subcover

Example

$\mathbb{R}_\ell \times \mathbb{R}_\ell$ ( basis) is not Lindelöf

Example

$\mathbb{R}$ is not compact

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} (n, n+2)$$

Claim

Let X be Hausdorff. If $C \subseteq X$ is compact then C is closed

Proof

Let $x \notin C$. If $x \notin \overline{C}$ $\Rightarrow C = \overline{C}$

X Hausdorff. Pick $U_y \ni x$, $V_y \ni y$, for each $y \in C$

$\Rightarrow C \subseteq \bigcup_{y \in C} V_y$ (as $y \in V_y$)

$\Rightarrow C$ compact

$\Rightarrow$ finite subcover

$$C \subseteq V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$$

$$U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n} \quad \text{open!}$$

$$U \cap V_{y_1} = \emptyset$$

$$\Rightarrow U \cap \left(\bigcup V_{y_i} \right) = \emptyset \Rightarrow U \cap C = \emptyset$$

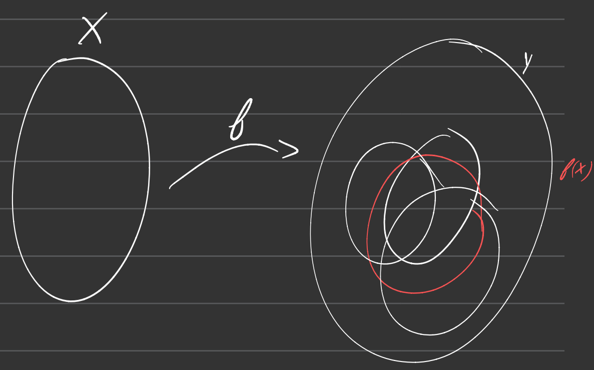
$$x \notin \overline{C} \,! \Rightarrow C = \overline{C} \Rightarrow C \text{ closed}$$

Fact

Let f be continuous. X compact
$\Rightarrow f(X)$ compact

Proof

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \quad , \quad U_\alpha \text{ open}$$



$f^{-1}(U_\alpha)$ is open

$\Rightarrow \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$ is

a union of open sets

$$= f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = f^{-1}(f(X))$$
$$= X$$

Let $X = f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n)$

be a finite subcover as X is compact

$$X = f^{-1}(U_1 \cup U_2 \cup \dots \cup U_n)$$

$$\Rightarrow U_1 \cup U_2 \cup \dots \cup U_n \supseteq f(X)$$

$\Rightarrow f(X)$ compact

Fact

Let f be a continuous map from a compact space to a Hausdorff space, f bijective. Then f is a homeomorphism

Proof

Let $C \subseteq X$ closed

X compact $\Rightarrow C$ compact

$\Rightarrow f(C)$ compact subset of Hausdorff subset, so closed

$\Rightarrow f$ closed map

$\Rightarrow f^{-1}$ continuous

$\Rightarrow f$ homeomorphism

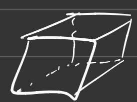
Compactness under Products

Fact

Let X be compact, Y compact
$\Rightarrow$ $X \times Y$ is compact

Example

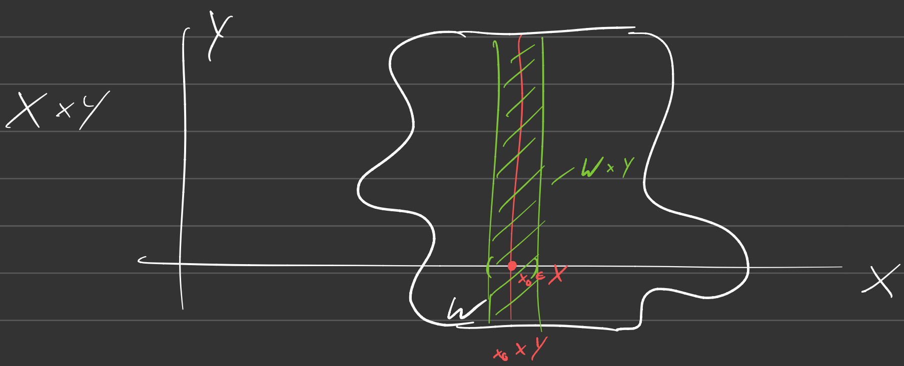
I compact $\Rightarrow$ $I \times I$ =  compact

I^3 =  compact

I^4 compact

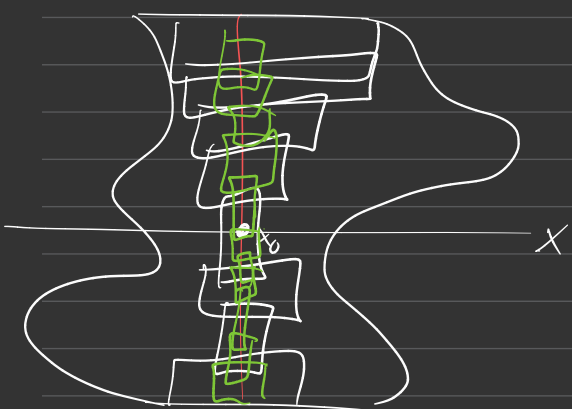
Lemma (Tube)

Let X, Y be compact. Let $N \subseteq X \times Y$ be open such that $x_0 \times Y \subseteq N$ for some x_0.



Then ~~there~~ exists a nbhd W of x_0

Proof



$x_0 \times Y = Y$

$(x_0, y) \mapsto y$

$\Rightarrow x_0 \times Y$ is compact (as Y is)

$\Rightarrow$ Pick a point $(x, y) \in x_0 \times Y$

$(x_0, y) \in N$ (open !)

$(x_0, y) \in U_y \times V_y \subseteq N$ $\quad (U_y \subseteq X,\ V_y \subseteq Y)$

$\Rightarrow$ get $(x_0, y) \in U_y \times V_y \subseteq N$ for all $y \in Y$

Consider $V_y \subseteq Y$ (open in Y)

$\bigcup V_y = Y \Rightarrow Y = V_{y_1} \cup V_{y_2} \cup \dots \cup U_{y_n}$ (finite subcover)

$y \in Y$

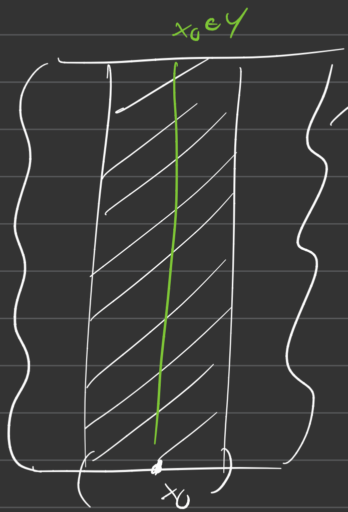
$\Rightarrow U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$

$\Rightarrow U \times Y \subseteq N$

$x_0 \in U_y \Rightarrow x \in W$

Compactness

Recap "Tube Lemma"



N open $\Rightarrow$ there exists a nbhd $W \subseteq X$ of $x \in X$ st $W \times Y \subseteq N$

tube

Claim

X, Y compact $\Rightarrow$ $X \times Y$ compact

Proof of Claim

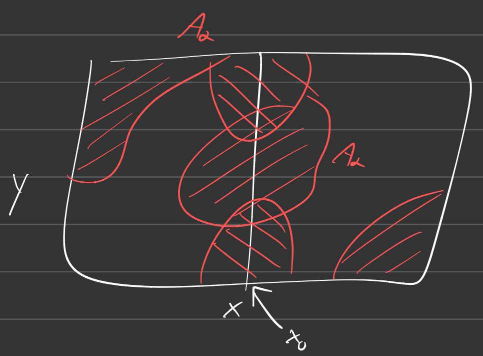
Let $X \times Y \subseteq \bigcup_{\alpha \in I} A_\alpha$

Let $x_0 \in X$ arbitrary

Then $x_0 \times Y \subseteq \bigcup_{\alpha \in I} A_\alpha \; (= x \times y)$

Y compact $\Rightarrow x_0 \times Y$ compact

$\Rightarrow x_0 \times Y \subseteq A_{\alpha_1} \cup A_{\alpha_2} \cup \dots \cup A_{\alpha_n}$

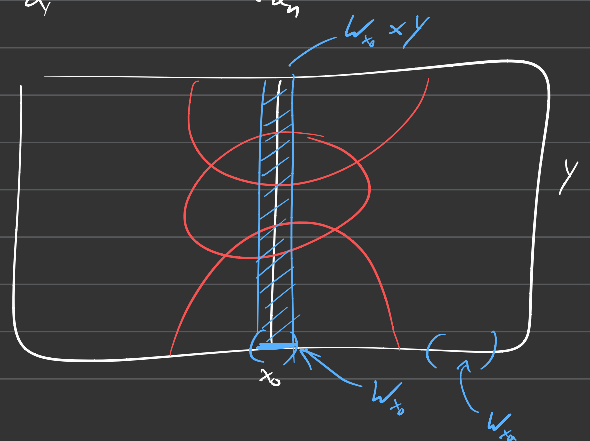


Apply tube Lemma to $N = A_{\alpha_1} \cup \dots \cup A_{\alpha_n}$

Since $x_0 \times Y \subseteq A_{\alpha_1} \cup \dots \cup A_{\alpha_n}$

the tube lemma gives us $x_0 \in W_{x_0}$ st

$W_{x_0} \times Y \subseteq A_{\alpha_1} \cup \dots \cup A_{\alpha_n}$



$$\bigcup_{x_0 \in X} W_{x_0} = X.$$

As X is compact there exists a finite subcover

$$X = W_{x_a} \cup \dots \cup W_{x_2} = \bigcup_i W_{x_i}$$

$$\Rightarrow X \times Y = \bigcup (W_{x_a} \times Y)$$

$$= \bigcup (A_{\alpha_1} \cup A_{\alpha_2} \cup \dots \cup A_{\alpha_n}) \subseteq \bigcup_{\alpha \in I} A_\alpha$$

$$= \left(\bigcup_i W_{x_i}\right) \times Y$$

$$= \bigcup_i (W_{x_i} \times Y)$$

Write $W_{x_i} \times Y \subseteq A_i^i \cup \dots A_n^i$

where $A_j^i = A_\alpha$ for some $\alpha \in I$

$$\Rightarrow X \times Y = \bigcup_i W_{x_j} \times Y$$

$$= \bigcup_i \left(\bigcup_j A_j^i \right) \subseteq \bigcup_{\alpha \in I} A_\alpha$$

$\Rightarrow$ finite subcover

Example

$[0,1]$ compact

$\Rightarrow$ $[0,1] \times [0,1]$ is compact

Connectedness

$S =$

$\subseteq \mathbb{R}$

Studying the topology of $S = (0,1) \cup (3,4]$ is essentially determined by the topology of $(0,1)$ and the topology of $(3,4]$

$(0,1) \subseteq S$ open $\quad (0,1)^c = S \setminus (0,1)$ open

Definition

A separation of a topological space $X = U \cup V$, U, V open, disjoint non empty subsets of X

If X admits a separation then X is disconnected

Fact

If X admits a separation, $X = U \cup V$ then U, V are open and closed

Example

S is disconnected

Fact

Suppose $X = U \cup V$ is a separation and $Y \subseteq X$ connected. Then $Y \subseteq U$ or $Y \subseteq V$

Proof

Write $Y = \underbrace{(U \cap Y)}_{\text{open in } Y} \cup \underbrace{(V \cap Y)}_{\text{open in } Y}$

Since Y does not admit a separation either $U \cap Y$ or $V \cap Y$ is empty

wlog $V \cap Y \neq \emptyset \Rightarrow Y = U \cap Y \subseteq U$

Fact

Let A be connected subspace of X. If B satisfies that $A \subseteq B \subseteq \bar{A}$ (is $B = A \cup \{\text{some limit points of } A\}$) then B is connected

Proof

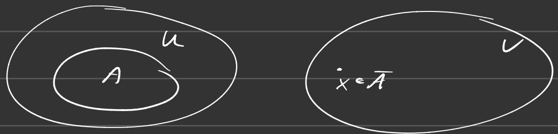
Suppose not. Write

$$B = U \cup V, \quad \text{a separation}$$

A connected $\Rightarrow A \subseteq U$ or $A \subseteq V$.
wlog assume $A \subseteq U$.

Let $x \in \bar{A}$. Suppose $x \in V$, then V is a nbhd of x st $V \cap A = \emptyset$
So $x \notin A$

$\Rightarrow$ contradiction



$\Rightarrow x \in \mathcal{U}$

$\Rightarrow \mathcal{B} \subseteq \mathcal{U}$

Theorem

The union of connected subspaces sharing a point is connected.

Proof

Suppose not

Let $Y = \bigcup_{\alpha \in I} Y_i$ where $x \in Y_i$, Y_i connected

Write $Y = U \cup V$, a separation. Then $Y_i \subseteq U$ (wlog)

$\Rightarrow x \in Y_i \subseteq U \Rightarrow x \in U \Rightarrow Y_i \subseteq U \quad \forall x \in Y_i$

$\Rightarrow \bigcup Y_i \subseteq U \Rightarrow Y \subseteq U \Rightarrow V = \emptyset$

Fact

If X is connected, $f: X \to Y$, f cts, then $f(X)$ is connected

Proof

Let $f(X) = U \cup V$, a separation. Then

$f^{-1}(U) \cup f^{-1}(V) = X$

is a separation a contradiction

Fact

X, Y connected $\Rightarrow X \times Y$ is connected

Proof

Let $T_{y_i} = (X \times \{y_i\}) \cup (\{x\} \times Y)$ for some fixed x.

$X \cong X \times \{y_i\}$ (homeomorphic)

and $Y \cong \{x\} \times Y$

$\Rightarrow (x, y_i) \in X \times \{y_i\}$

and $(x, y) \in \{x\} \times Y$

$\Rightarrow T_{y_i}$ connected

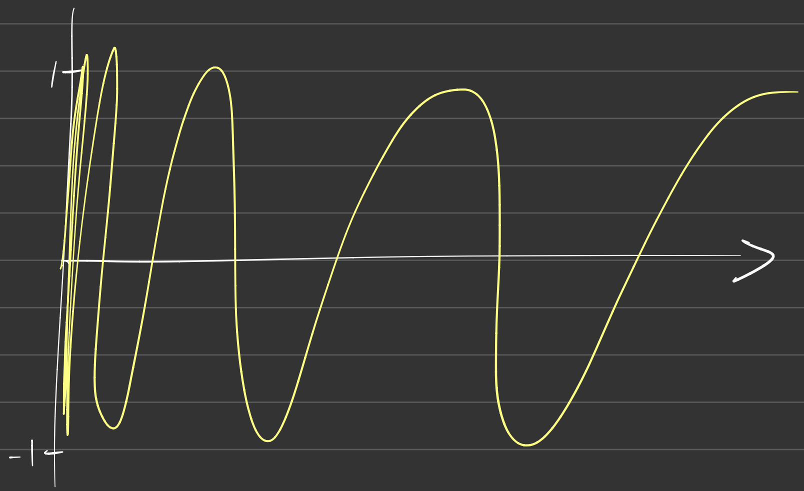
$\Rightarrow X \times Y = \bigcup_{y \in Y} T_{y_i}$ is connected

as $(x, y) \in T_{y_i} \quad \forall i$

Topologists Sine Curve

Let $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$
$x \longmapsto \sin(\frac{1}{x})$

Consider the graph of $f \cup \{0\} \times [-1, 1]$



Claim

Topologists sine curve is connected

Proof

(1) First we show that the graph of f is connected (image of connected set under continuous map g)

$g(x) = (x, \sin(\frac{1}{x}))$

(2) B is a set st $A \subseteq B \subseteq \bar{A}$, A connected
$\Rightarrow B$ connected

Start

Let $(x,y) \in \{0\} \times [-1, 1]$, claim $\exists$ a sequence

$(v_n) \subseteq \text{graph}(f)$

such that $\quad v_n \longrightarrow (x,y) \Rightarrow (x,y) \in TSC$

Let $\quad r \in [-1, 1]$

Then by intermediate value theorem $\exists$
$x > 0 \quad$ st $\quad \sin\left(\frac{1}{x}\right) = r$

$\Rightarrow \sin\left(\frac{1}{x} + 2k\pi\right) = r \qquad$ for $k \in \mathbb{Z}$

$\Rightarrow \quad \frac{1}{x'} = \frac{1}{x} + 2k\pi$

$\Rightarrow \quad x' = \dfrac{1}{\frac{1}{x} + 2k\pi}$

$\Rightarrow \lim_{k \to \infty} \dfrac{1}{\frac{1}{x} + 2k\pi} = 0$

$\Rightarrow (0, r)$ is the limit point of

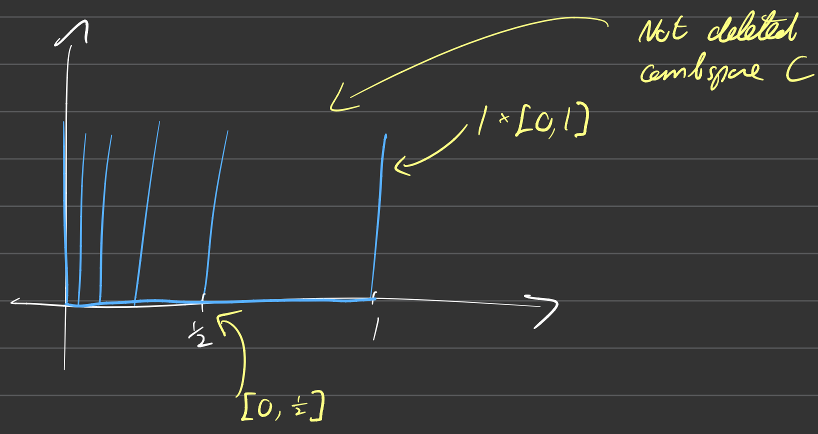
$$v_k = \left(\frac{1}{\frac{1}{x} + 2k\pi}, \sin\left(\frac{1}{\frac{1}{x} + 2k\pi} \right) \right)$$

$\Rightarrow v_k \to (0, r)$

$\Rightarrow (0, r) \in \overline{\text{graph}(f)}$ for any $r \in [-1, 1]$

$\Rightarrow TSC = \overline{\text{graph}(f)} \Rightarrow$ connected

The Comb Space



The space is

$$[0,1] \times \{0\} \cup \{ \{\tfrac{1}{n}\} \times [0,1] \mid n \in \mathbb{N} \} \cup \{0\} \times [0,1]$$

Deleted combspace is

$$C \setminus \{(0,0), (0,1)\}$$

Claim

C is connected

Proof

Connected? by previous result, C is a union of ~~these~~ sets containing $(0,0)$

Lemma

X connected $\Leftrightarrow$ a continuous function $f: X \to \{0,1\}$ cannot be surjective ← discrete

Proof

$(\Rightarrow)$ X connected, f cts

$\Rightarrow$ $f(X)$ is connected

But $\{0\} \cup \{1\}$ form a separation of $\{0,1\}$

$\Rightarrow$ $f(X) \subseteq \{0\}$ or $f(X) \subseteq \{1\}$

$\Rightarrow$ cannot be surjective

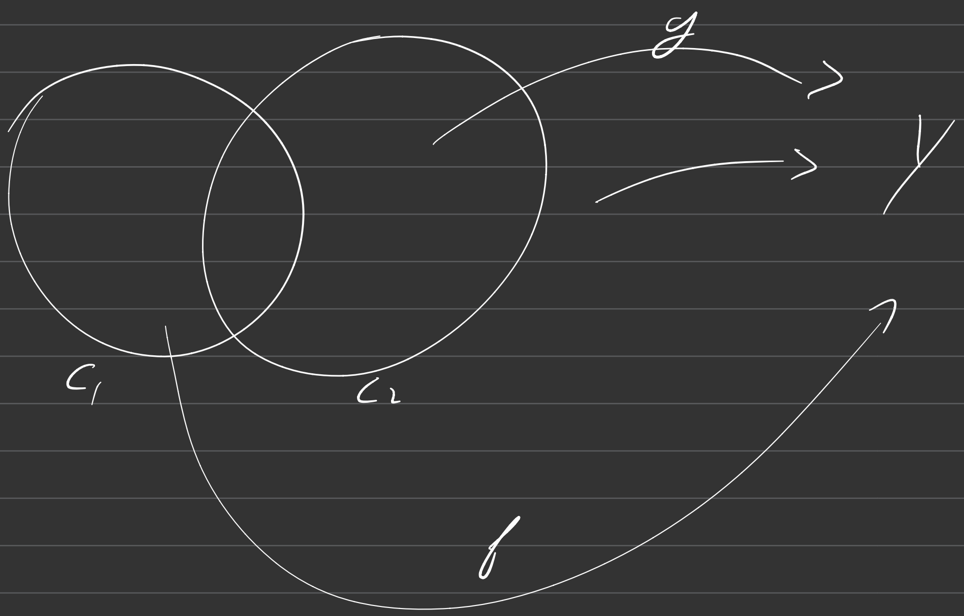
(=>) Pasting Lemma: $f: C_1 \to Y$ cts $g: C_2 \to Y$ cts, $C_1, C_2 \subseteq X$ are closed

$$f|_{C_1 \cap C_2} = g|_{C_1 \cap C_2}$$

$\Rightarrow$ $h: C_1 \cap C_2 - Y$ given by

$h(1) = f(c)$, $c \in C_1$, $h(c) = g(c)$ $c \in C_2$

is cts



We will apply this lemma as follows

Let $U \cup V$ be a separation of X
U, V are closed !

$h : X \to \{0,1\}$ as

$$x \mapsto \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$$

This is cts by the pasting lemma
as

$f : U \to \{0,1\}$ $x \mapsto 0$ and

$g : V \to \{0,1\}$ $x \mapsto 1$

are continuous

$U \cap V = \emptyset$, U, V closed

$\Rightarrow$ Pasting Lemma

$\Rightarrow$ h cts

$\Rightarrow$ contradiction

Definition

Let X be a top space. Define $x \sim y$ if there exists a connected set C st $x, y \in C$. This is an equivalence relation (omitted).

Definition

The equivalence classes are called connected components

Connected components

$[0,1]$, $[2,3)$

Example

$\mathbb{R}$ is connected, so the connected components is $\mathbb{R}$

Example

$\mathbb{N} \subseteq \mathbb{R}$. The connected components are the singletons

Example

$\mathbb{Q} \subseteq \mathbb{R}$ same

Fact

Connected components may not be open
($\{a\}$ not open $\forall$ in)

Theorem

Connected components are closed and connected.

Proof

(1) connected.

Let C be a connected component

Pick some $x \in C$

Let $y \in C$ be arbitrary. Since $x, y \in C$
$\iff x \sim y \iff \exists$ a connected set
C_y st $x, y \in C_y$. Then $C_y \subseteq C$ (if
not $z \in C_y$ st $z \notin C \Rightarrow$ but then
$x \sim z \Rightarrow z \in C$)

$\Rightarrow C = \bigcup_{y \in C} C_y$, a union of connected sets all of which contain x

$\Rightarrow$ C is connected as a union of connected sets sharing a point

(2) <u>closed</u>

A connected $\Rightarrow \overline{A}$ connected

$[x] = C$ be a connected component
Let $y \in \overline{C}$. Then $\overline{C}$ is a connected set (as C is connected) so $y \sim x$
(as $y, x \in \overline{C}$ connected) $\Rightarrow y \sim x \Rightarrow y \in C$

<u>Definition</u>

X is disconnected if X is not connected.

<u>Characterisation</u>

Adjoining a point to a connected component yields a disconnected set

Fact

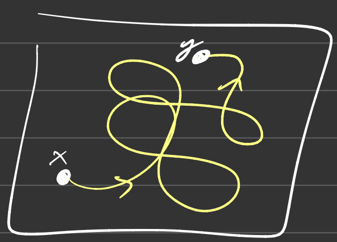
A connected subset is contained in a connected component

Path Connectedness

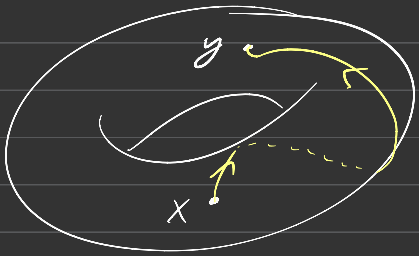
Definition

A space is path connected if $\forall x, y \in X$ there exists $\gamma : I \to X$, cts st $\gamma(0) = x$, $\gamma(1) = y$

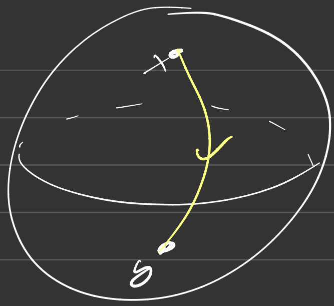
Examples

- 

$I \times I$ is path connected

- 

$T^2 = S^1 \times S^1$ is path connected



Fact

X path connected ⇒ X connected

Example

Topologist's sin curve, NOT path connected

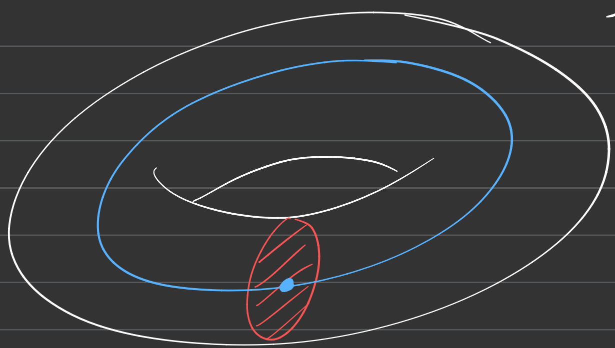
Homotopy

Definition

A deformation retract $r_t : X \to A$, where $A \subseteq X$ is a map $r : X \times I \to X$ such that

$$r_t|_A = id \qquad r_0 = id|_X$$

Solid Torus

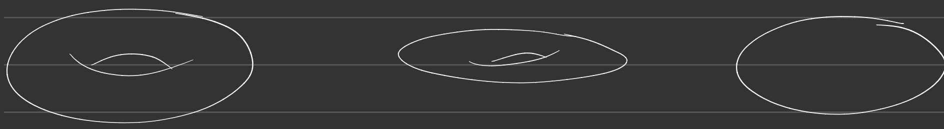


Example

(1) $X = D^2 \times S^1$ $\qquad r_t(p, s)$

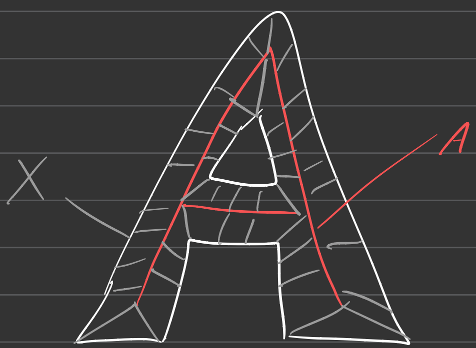
$A = \{0\} \times S^1 \qquad = ((1-t)p, s)$

$S^1 \times D^2 =$

 $= S^1$

$t=0$ $t=1$

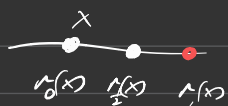
(2)



<u>Claim</u>

X def retracts to A

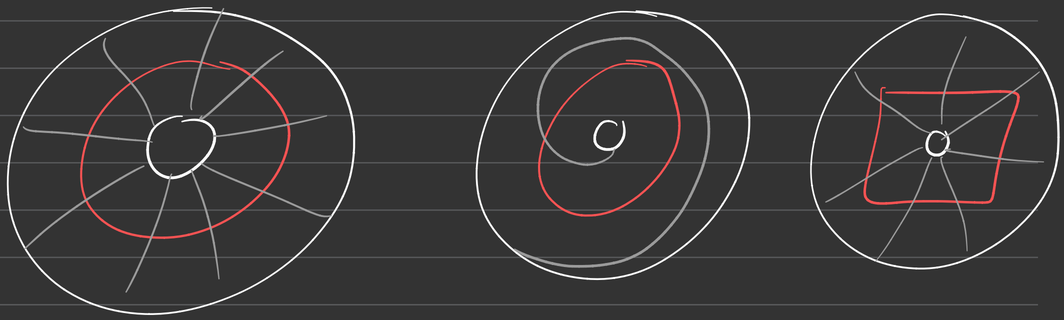
Proof



$\longleftarrow$ describes $r_t(x)$ $\forall x \in X$

Fact

Deformation retract



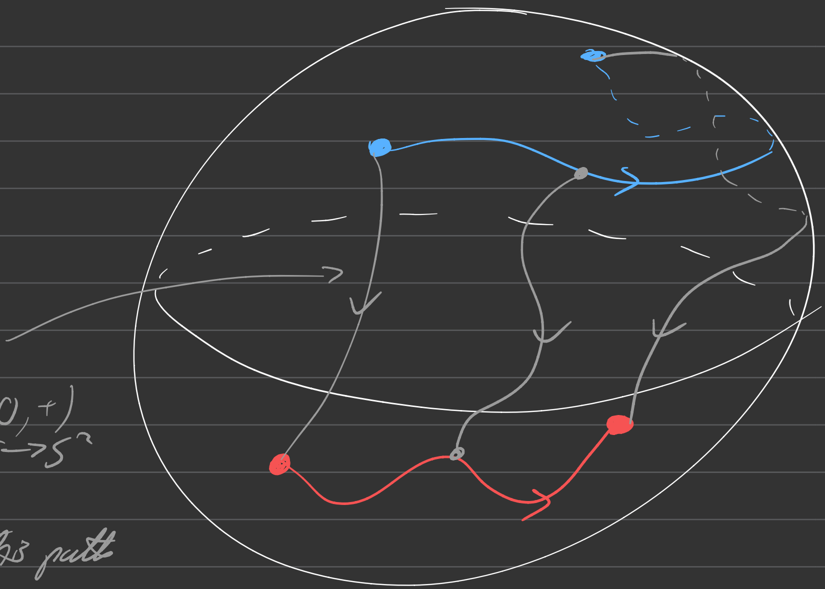
Definition

Let $f, g : X \longrightarrow Y$ cts. We say that f, g are homotopic if there exists

$H : \underbrace{X \times I}_{\text{product topology}} \longrightarrow Y$ st H cts

$H(-,0) = f, \quad H(-,1) = g$

$\alpha : I \longrightarrow S^2$



$H:(0,t)$
$: I \to S^2$

is this path

$\gamma : I \to S^2$

<u>claim</u> $\alpha \simeq_h \gamma$ (α homotopic to γ)

need $H : I \times I \longrightarrow S^2$

st $H(-,0) = \alpha$

$H(-,1) = \gamma$

Definition

X, Y are homotopy equivalent if there exists $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ st

(1) $f \circ g \simeq_h id_Y$

(2) $g \circ f \simeq_h id_X$

Fact

If X, Y are homeomorphic then they are homotopy equivalent.

Example

$\mathbb{R}$ and $\{0\}$ are homotopy equivalent (but not homeomorphic)

Need $f: \mathbb{R} \longrightarrow \{0\}$

$g: \{0\} \longrightarrow \mathbb{R}$

Proof

Need to prove

$f \circ g \simeq id_{\{0\}}$ (only one map $\{0\} \to \{0\}$)

$g \circ f : \mathbb{R} \to \mathbb{R}$

$g(f(x)) = g(0) = 0$

To show $g \circ f \simeq id_{\mathbb{R}}$

is $\exists$ $H : \mathbb{R} \times I \to \mathbb{R}$

st $H(-, 0) = id$

$H(-, 1) = g \circ f = x \mapsto 0$

$H(x, t) = (1-t)x$ cts

$H(x, 0) = (1-0)x = x$

$H(x, 1) = 0$

$\Rightarrow \mathbb{R}$ and $\{0\}$ are homotopy equivalent

Fact

If X deformation retracts to A then X and A are homotopy equivalent

Proof

We need $f: X \to A$
$g: A \to X$ st

$$f \circ g \simeq id_A \;,\; g \circ f \simeq id_X$$

g = subspace inclusion

$$g(a) = a$$

As X deformation retracts to A then we have

$$r_t : X \to X \quad st$$

$$r_0 = id_X \;,\; r_1(X) \in A, \quad r_t|_A = id$$

$$f \circ g(a) = r_1(g(a)) = r_1(a) = a$$

$$\Rightarrow f \circ g \simeq_h id$$

$$g \circ f(x) = g(\underbrace{r_1(x)}_{\in A}) = r_1(x)$$

$\Rightarrow g \circ f = r_1 \Rightarrow$ need homotopy from r_1 to r_0

The deformation retract is

$H : X \times I \to X$

is a cts map st

$H(-,0) = r_0$, $H(-,1) = r_1$

$\Rightarrow r_0$ and r_1 are homotopic

The converse is not true

X, A st $X \simeq A$

but X does not def retract to A

$\{0\}, \{1\}$

$\simeq = \{0, 1\}$

Fact

If X, Y are homotopy equivalent then $\exists$ Z st $X \subseteq Z$, $Y \subseteq Z$ and Z def retracts to both X and Y

Claim

Homotopy equivalence is an equivalence relation

(1) $X \simeq Y$

(2) $X \simeq Y \Rightarrow Y \simeq X$

(3) $X \simeq Y,\ Y \simeq Z \Rightarrow X \simeq Z$

Proof

(3)
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "u"] & Z \arrow[l, bend left, "l"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[rr, bend left=60, "k \circ f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "k"] & Z \arrow[l, bend left, "l"] \arrow[ll, bend left=60, "g \circ l"] \end{tikzcd}$$

$$\Rightarrow (k \circ f) \circ (g \circ l) \simeq id_X$$

$$(g \circ l) \circ (k \circ f) \simeq id_Z$$

Fact

Homotopy equivalence is an equivalence relation

X, Y homotopy equivalent

$$\Leftrightarrow \begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

st $f \circ g = id_Y$, $g \circ f \simeq id_X$

ie there exists a map

$H(-,0) = f \circ g$

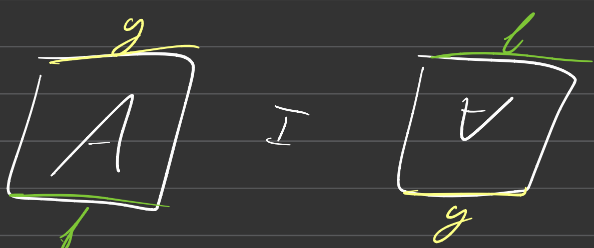
$H : Y \times I \longrightarrow Y$ st $H(-,1) = id$

Fact

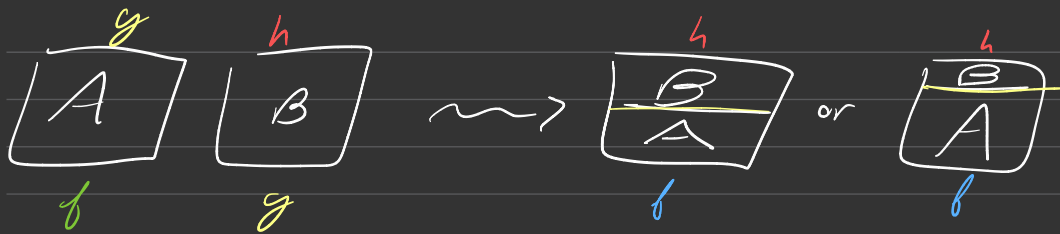
$f \sim g \iff f$ homotopic to g
is an equivalence relation

(1) $f \sim f$

(2) $f \sim g \Rightarrow g \sim f$



(3) $f \sim g,\ g \sim h \Rightarrow f \sim h$



$$X \sim Y,\quad Y \sim Z \implies X \sim Z$$

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] & Y \arrow[l, bend left] \arrow[r, bend left, "k"] & Z \arrow[l, bend left, "l"]
\end{tikzcd}$$

$$H^y : Y \times I \longrightarrow Y$$

$$H^y(-,0) = l \circ k$$

$$H^y(-,1) = id$$

To prove $(g \circ l) \circ (k \circ f) = id_y$

$$H : X \times I \longrightarrow X$$

$$H(x,t) = g\, H^y(f(x), t)$$

$H(x,0) = g H^Y(f(x),0)$

$= (g \circ (l \circ k) \circ f)(x)$

$= (g \circ l) \circ (k \circ f)$

$H(x,1) = g H^Y(f(x),1) = g(f(x))$

$= g \circ f(x)$

$\Rightarrow H(-,0) = (g \circ l) \circ (k \circ f)$

$H(-,1) = g \circ f$

$\Rightarrow g \circ l \circ k \circ f \sim g \circ f \simeq_h id_X$

$\Rightarrow g \circ l \circ k \circ f \simeq_h id_X$

Fact

X, Y are homotopy equivalent

$\Leftrightarrow$ $\exists$ a Z st Z def retracts to X, also def retract to Y